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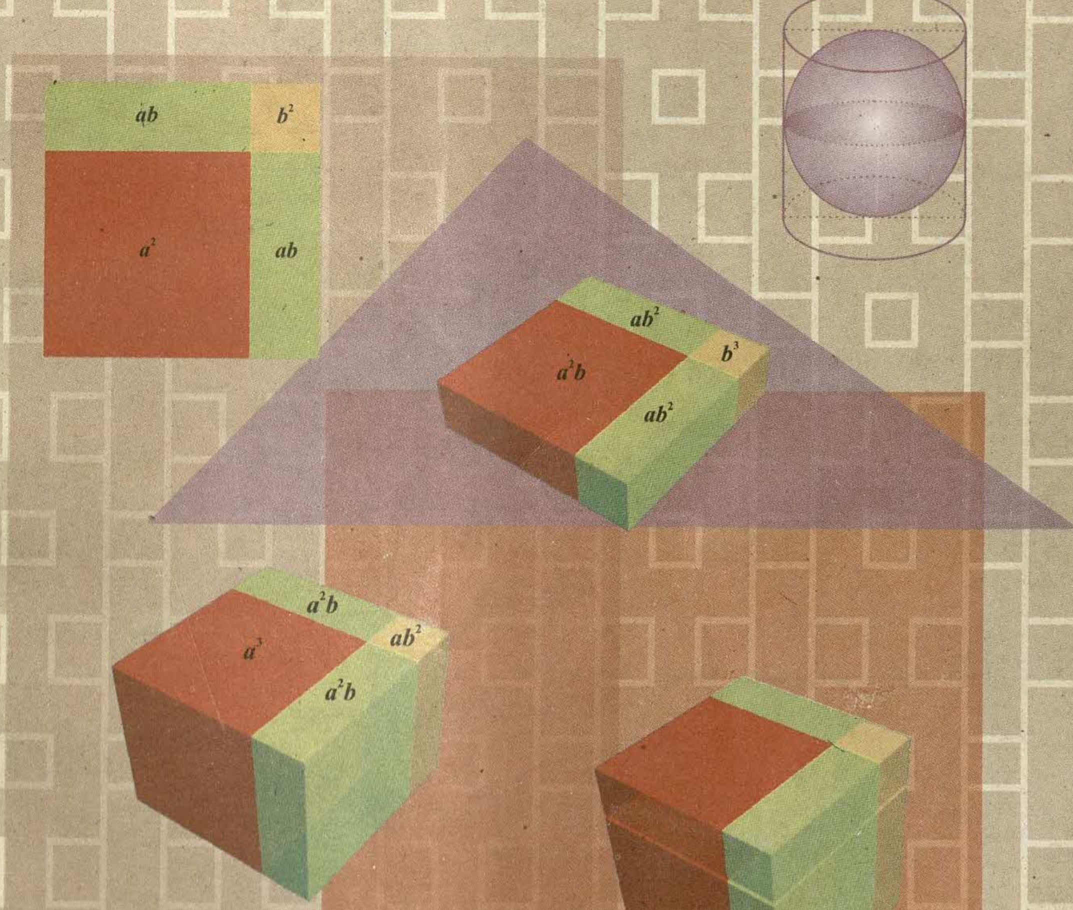
MATHEMATICS

Textbook for Class VIII



$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

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TEXTBOOK FOR CLASS VIII

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विद्ययाऽमृतमश्नुते

एन सी ई आर टी
NCERT

राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्

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PUBLISHER'S NOTE

The National Council of Educational Research and Training (NCERT) has been preparing and publishing school textbooks and other educational material for children and teachers. These publications are regularly revised on the basis of feedback from students, teachers, parents, and teacher educators. Research done by the NCERT also forms the basis for updating and revision.

This book is based on the National Curriculum Framework for School Education – 2000 and the syllabi prepared in accordance with it. The Executive Committee of the NCERT, in its meeting held on 19 July 2004, discussed all aspects related to the quality of textbooks and decided that the textbooks of all subjects should undergo a quick review. In pursuance of this decision, the NCERT constituted 23 Quick Review Committees to examine all the textbooks. These committees identified various errors of conceptual, factual and linguistic nature. The review process also took note of the evaluation of textbooks undertaken earlier. The exercise has now been completed and the errors identified have been corrected. We hope that this revised edition will serve as an effective medium of teaching and learning. We look forward to your suggestions to enable us to further improve the quality of this book.

SECRETARY
National Council of Educational
Research and Training

New Delhi
January 2005

CONSTITUTION OF INDIA

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a **SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC** and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

PREFACE

Mathematics has been an inseparable part of school education since the beginning of formal education and it has played a predominant role not only in the advancement of civilizations but also in the development of physical sciences and other disciplines. As curriculum renewal is a continuous process, Mathematics curriculum has undergone various types of changes from time to time in accordance with the changing needs of the society. The present effort of reforming and updating the curriculum in Mathematics at the Upper Primary Stage is an exercise based on the feedback from users, emergence of new vistas of knowledge and various curricular concerns given in the *National Curriculum Framework for School Education (NCFSE)* brought out by the National Council of Educational Research and Training (NCERT) in November 2000 after in-depth discussions. The Draft Discussion Document on Curriculum Framework was prepared earlier by various teacher educators, nominees from various examination boards, representatives of the Directorates of Education and the State Councils of Educational Research and Training (SCERTs) of different States/Union Territories, general public and faculty from the universities, colleges, schools and the Council.

Some of the general curricular concerns that have emerged from the NCFSE–2000 relevant to teaching of Mathematics at the Upper Primary Stage are as follows :

- Creating an awareness of equality of all with a view to removing prejudices transmitted through social environment and the factor of birth.
- Education for girls.
- Protection of environment.
- Integration of indigenous knowledge and India's contribution to Science and Mathematics from ancient times till date.
- Reduction in the curriculum load by taking out obsolete and redundant contents and providing vertical linkage with Mathematics to be taught at the Secondary Stage.

Keeping in view the above concerns, NCERT constituted writing teams for developing Mathematics textbooks for different stages of schooling. The Writing Team for the Upper Primary Stage has already developed textbooks of Mathematics for Classes VI and VII. The present textbook in Mathematics for Class VIII is next in this series.

A lot of effort has gone into the preparation of the book. First, the drafts prepared by various authors were discussed amongst the team members and modified in the light of the comments and observations made. The modified material was exposed to a group of teachers and experts in a Review Workshop. The manuscript of the book was finalised keeping in view the comments and suggestions made by the participants of the Review Workshop.

The salient features of the textbook are as follows :

- As far as possible, each topic has been introduced through motivating examples relating to the immediate environment of the pupils.
- The book contains detailed explanations of concepts and a large number of illustrations, solved examples and exercises so that the pupils may grasp the fundamentals on the one hand and may acquire the desired problem-solving and learning skills on the other.
- A number of learning activities like paper cutting/assembling and *Do it with your friends* have been suggested for the pupils to rediscover the relevant Mathematical facts, and to develop appropriate drawing and measuring skills.
- Some of the included word problems develop awareness towards the need for national integration, protection of environment, removal of social barriers, observance of small family norms, elimination of sex-bias and so on. It is expected that the message of these problems would get into the minds of the pupils, and the teachers would be conscious of this while teaching.
- The vocabulary and terminology used in the book is in accordance with the comprehension and maturity level of the pupils.
- A list of important concepts and results has been provided at the end of each Chapter in the form of *Things to Remember.*
- Historical references, especially to Indian contributions, have been provided at the end of each Unit under the title *As History Tells Us.*

I am thankful to the Director, NCERT, who initiated this challenging activity of tremendous national importance and gave us a chance to return the debt that we owe to our profession towards the improvement of Mathematics education. I am

also thankful to the Head, DESM, for his dynamic leadership, enthusiastic cooperation and generation of facilities as and when required. It goes without saying that my warmest thanks are due to the members of the Writing Team, and the participants of the Review Workshop. I am also thankful to the members of CIET who provided help with some diagrams.

To put a full stop to this somewhat lengthy *Preface*, I must mention the too often repeated warning that no book on any subject can ever be the last word on the subject. We have tried to do our best during the limited time that we had at our disposal, and are conscious that improvements are possible. Suggestions/ comments/observations for the improvement are most welcome. I hope the readers would get as much pleasure in reading the textbook as we have got in writing the same.

ASHA RANI SINGAL
Chairperson

CONSTITUTION OF INDIA

Part IV A (Article 51 A)

Fundamental Duties

Fundamental Duties – It shall be the duty of every citizen of India —

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wildlife and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(k) who is a parent or guardian, to provide opportunities for education to his child or, as the case may be, ward between the age of six and fourteen years.

ACKNOWLEDGEMENTS

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The Council acknowledges the valuable contributions of the following participants of the Review Workshop in the finalisation of this book : Professor G.P. Dikshit, Department of Mathematics and Astronomy, Lucknow University, Uttar Pradesh; Smt. Aliya Parveen, *TGT,* C.R.P.F. Public School, Jawahar Nagar, Secunderabad, Andhra Pradesh; Shri Ashok Kumar Gupta, *TGT,* Sarvodaya Vidyalaya, Pitampura, Delhi; Daljeet Kaur Sethi, *TGT,* Delhi Public School, Vasant Kunj, New Delhi; Shri N.D. Saini, *TGT*, Kendriya Vidyalaya, Gurgaon, Haryana; Smt. N. Sahni, *TGT,* B.S.F. Senior Secondary School, Sriganganagar, Rajasthan; Shri N. Nandal, *TGT,* Sainik School, Kunjpura, Karnal, Haryana; Shri P. K. Tewari, *Assistant Commissioner* (*Retd.*), Kendriya Vidyalaya Sangathan, New Delhi; Smt. Rajni Sharma, *TGT,* Ahlcon Public School, Mayur Vihar, Phase I, New Delhi; Smt. Reena Raghav, *TGT,* DAV Public School, Sector – 14, Gurgaon, Haryana; Ms. Ruchi Salaria, *TGT,* Kendriya Vidyalaya No. 4, Delhi Cantt.; Smt. S. Bhuvana, *TGT,* D.T.E.A. Senior Secondary School, R.K. Puram, New Delhi; Smt. Sankar Misra, *TGT,* Demonstration Multipurpose School, RIE, Bhubaneswar, Orissa; Smt. Sarita Rewri, *TGT,* Government Girls Senior Secondary School No.1, Roop Nagar, Delhi; Professor Satya Narain Chaurasia, S.I.S.E., Allahabad, Uttar Pradesh; Smt. Ved Dudeja, *Vice Principal,* Government Co-Educational Secondary School, Sainik Vihar, New Delhi and Professor Hukum Singh; Dr. V. P. Singh, *Reader;* Shri P. K.Chaurasia, *Lecturer* from NCERT.

The Council also gratefully acknowledges the feedback and suggestions of the following members of the Quick Review Committee for the improvement of this book : Professor R. Bhatia, (*Chairperson*) Indian Statistical Institute, New Delhi; Professor Pravin Sinclair, School of Sciences, IGNOU, New Delhi; Dr. Geetha Venkataraman, Department of Mathematics, St. Stephen's College, University of Delhi; Professor P. Kaul, IASE, Jamia Millia Islamia, New Delhi; Dr. R.P. Maurya,

Reader, RIE, Ajmer; Shri Mahendra Shanker, *Lecturer* (S.G.) (*Retd.*), NCERT, New Delhi; Smt. Anju Bhatia, *TGT,* Sarvodaya Kanya Vidyalaya, Naraina, New Delhi; Smt. Mala Mani, *TGT,* Amity International School, Noida; Shri Sunil Aggarwal, *TGT,* Government Sarvodaya Co-Ed. Senior Secondary School, Janakpuri, New Delhi and Dr V.P. Singh, *Reader;* Dr. A.K. Wazalwar, *Reader*; Shri Praveen Kumar Chaurasia, *Lecturer* (*Coordinator*) from NCERT, New Delhi.

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CONSTITUTION OF INDIA

Part III (Articles 12 – 35)
(Subject to certain conditions, some exceptions and reasonable restrictions)

guarantees these

Fundamental Rights

Right to Equality

- before law and equal protection of laws;
- irrespective of religion, race, caste, sex or place of birth;
- of opportunity in public employment;
- by abolition of untouchability and titles.

Right to Freedom

- of expression, assembly, association, movement, residence and profession;
- of certain protections in respect of conviction for offences;
- of protection of life and personal liberty;
- of free and compulsory education for children between the age of six and fourteen years;
- of protection against arrest and detention in certain cases.

Right against Exploitation

- for prohibition of traffic in human beings and forced labour;
- for prohibition of employment of children in hazardous jobs.

Right to Freedom of Religion

- freedom of conscience and free profession, practice and propagation of religion;
- freedom to manage religious affairs;
- freedom as to payment of taxes for promotion of any particular religion;
- freedom as to attendance at religious instruction or religious worship in educational institutions wholly maintained by the State.

Cultural and Educational Rights

- for protection of interests of minorities to conserve their language, script and culture;
- for minorities to establish and administer educational institutions of their choice.

Right to Constitutional Remedies

- by issuance of directions or orders or writs by the Supreme Court and High Courts for enforcement of these Fundamental Rights.

CHAPTER

1

SQUARES AND SQUARE ROOTS

1.1 Introduction

In earlier classes, we have studied numbers which can be obtained by raising rational numbers to integral exponents. When the exponent is two, the numbers obtained are called *squares* or *square numbers* and the number itself is called the *square root* of the square so obtained. In this Chapter, we will study *square numbers* and their *square roots*. We first discuss a few patterns of square numbers and then describe some simple methods for obtaining squares of two and three digit numbers. We then study square roots of perfect square numbers (i.e., squares of integers) through prime factorisation method. We also study the *divison method* for finding the square roots of perfect square numbers, rational numbers and decimal numbers. If a number is not a perfect square, we cannot find its square root in the form of an integer. However, we may try to find a fraction or a decimal number whose square is approximately equal to the given number. Such numbers are called approximate square roots. We use the division method for this purpose.

1.2 Square of a Number and Square Numbers

If m and n are natural numbers such that $n = m^2$, then n is the square of the number m and the number n is a *square number*. For example, 4 ($= 2 \times 2 = 2^2$) is the square of 2, and is therefore, a square number. In Table 1.1, given at the end of the section, we give squares of all natural numbers upto 20, and consequently a list of square numbers upto 400. The square of an integer is also called a *perfect square* (or a *perfect second power*).

We can see that not all natural numbers are perfect squares. Upto 100, only ten numbers are perfect squares (Table 1.1). Upto 10000, only hundred numbers are perfect squares. Numbers like 3, 50, 1700 are not perfect squares or square numbers.

Some problems that need attention in connection with square numbers are as follows :

1. To determine *whether a given number is a perfect square.*
2. Given a perfect square, *to determine the number of which it is a square.*

These problems have important applications in many areas of science and technology. To understand the basic facts related with these problems in a better way, we first observe some properties and patterns of numbers which are perfect squares, and also of those numbers which are not perfect squares.

Table 1.1: Squares of 1 to 20

Number	*Square*	*Number*	*Square*
1	1	11	121
2	4	12	144
3	9	13	169
4	16	14	196
5	25	15	225
6	36	16	256
7	49	17	289
8	64	18	324
9	81	19	361
10	100	20	400

1.3 Properties and Patterns of some Square Numbers

I. *No square number ends in* 2, 3, 7 *or* 8. A glance at Table 1.1 shows that perfect square numbers end in 0, 1, 4, 5, 6 or 9. There is nothing special about numbers from 1 to 20. The square of any number would end in any of these digits only. Take a number n having more than one digit and square it. Now take the units digit of n, square it and take its units digit. This units digit is the same as the units digit of n^2. In other words, the square of any number would end in 0, 1, 4, 5, 6 or 9. It means that numbers ending in 2, 3, 7 or 8 are not perfect squares. Consequently, none of the numbers 52, 793, 15857, 888888 is a perfect square. Do we also conclude that numbers ending in 0, 1, 4, 5, 6 or 9 are necessarily perfect squares? How about 10, 11, 14, 15, 26, 19 ?

Remark : We shall use the phrase *number ends in a* to mean that the *units digit of the number is a.*

II. *Given the units digit of a number, we can determine the units digit of its square*. If a number ends in 1 or 9, its square would end in 1. The square of a number which ends in 2 or 8, would end in 4. The square of a number ending in 3 or 7 would end in 9. Similarly, if 4 or 6 is the units digit of a number, the units digit of its square would be 6. If a number ends in 5 or 0, its square would end in 5 or 0, respectively.

III. *The number of zeros at the end of a perfect square is always even*. If a number ends in a zero, (i.e., 50), its square (2500) ends in double zero. In fact, the number of zeros at the end of the square of a number is twice the number of zeros at the end of the number, e.g., $90000 = 300^2$. Therefore, *a number ending in an odd number of zeros is never a perfect square*.

IV. *If a number is even (odd), then so is its square*. This fact can be easily verified from Table 1.1 for numbers upto 20. However, the result is true in general and follows from Property II. (Since even numbers end in 0, 2, 4, 6, 8 and odd numbers end in 1, 3, 5, 7, 9.)

V. *A perfect square leaves a remainder* 0 *or* 1 *on division by* 3. This can be checked by dividing square numbers in Table 1.1 by 3, but the result is true for all square numbers. The following table gives the possible remainders for some other prime divisors.

Table 1.2

Divisor	*Possible Remainders*
3	0, 1
5	0, 1, 4
7	0, 1, 2, 4
11	0, 1, 3, 4, 5, 9
13	0, 1, 3, 4, 9, 10, 12

VI. The list of possible remainders in V above can be helpful in checking when a number is *not* a perfect square. For example, if we divide a number by 3 and get the remainder 2, then the number is *not* a perfect square.

VII. *If n is a perfect square, then* $2n$ *can never be a perfect square*. In other words, if $n = q^2$ for some natural number q, then we cannot find a natural number p such that $2q^2 = p^2$. This fact can be verified for numbers less than 200 using Table 1.1. It is true in general. In fact, we can show that if *t is a prime and n is a perfect square, then tn is not a perfect square*.

VIII. The squares of numbers like 1, 11, 111, . . . etc., which are composed of digit 1 alone, have a nice pattern as shown below :

$$1^2 = 1$$
$$11^2 = 1\,2\,1$$
$$111^2 = 1\,2\,3\,2\,1$$
$$\vdots$$
$$111111111^2 = 1\,2\,3\,4\,5\,6\,7\,8\,9\,8\,7\,6\,5\,4\,3\,2\,1$$

IX. We have another interesting pattern related to squares of numbers :

$$1^2 = 1$$
$$11^2 = 1\,2\,1, \text{ and } 1 + 2 + 1 = 2^2$$
$$111^2 = 1\,2\,3\,2\,1, \text{ and } 1 + 2 + 3 + 2 + 1 = 3^2$$
$$\vdots$$
$$111111111^2 = 1\,2\,3\,4\,5\,6\,7\,8\,9\,8\,7\;6\,5\,4\,3\,2\,1 \text{ and } 1 + 2 + \ldots + 2 + 1 = 9^2$$

X. Yet another pattern is given below :

$$121 \times (1 + 2 + 1) = 484 = 22^2$$
$$12321 \times (1 + 2 + 3 + 2 + 1) = 110889 = 333^2$$

i.e.,
$$11^2 \times (\text{sum of digits in } 11^2) = 22^2$$
$$111^2 \times (\text{sum of digits in } 111^2) = 333^2$$
$$\vdots$$
$$111111111^2 \times (\text{sum of digits in } 111111111^2) = 999999999^2$$

EXERCISE 1.1

1. The following numbers are not perfect squares. Give reason.
   (i) 1057 (ii) 23453 (iii) 7928 (iv) 222222
2. What will be the units digit of the squares of the following numbers?
   (i) 81 (ii) 272 (iii) 799 (iv) 3853
   (v) 1234 (vi) 26387 (vii) 52698 (viii) 99880
   (ix) 12796 (x) 55555
3. The following numbers are not square numbers. Give reason.
   (i) 64000 (ii) 89722 (iii) 222000 (iv) 505050
4. The square of which of the following would be an odd number ?
   (i) 431 (ii) 2826 (iii) 7779 (iv) 82004
5. Show that the following numbers are not perfect squares :
   (i) 7927 (ii) 1058 (iii) 33453 (iv) 22222
   [*Hint* : Use Property V]
6. Observe the following pattern and find the missing digits :

$$11^2 = 121$$
$$101^2 = 10201$$
$$1001^2 = 1002001$$
$$100001^2 = 1\ldots\ldots2\ldots\ldots1$$
$$10000001^2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

7. Observe the following pattern and supply the missing numbers :

$$11^2 = 121$$
$$101^2 = 10201$$
$$10101^2 = 102030201$$
$$1010101^2 = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots^2 = 10203040504030201$$

8. Using the given pattern, find the missing numbers :

$$1^2 + 2^2 + 2^2 = 3^2$$
$$2^2 + 3^2 + 6^2 = 7^2$$
$$3^2 + 4^2 + 12^2 = 13^2$$

$4^2 + 5^2 + \_\_^2 = 21^2$

$5^2 + \_\_^2 + 30^2 = 31^2$

$6^2 + 7^2 + \_\_^2 = \_\_^2$

9. Using suitable patterns, complete the following :

(i) $\dfrac{333^2}{12321} = \text{—}$ (ii) $\dfrac{666666^2}{12345654321} = \text{—}$

10. Write true (T) or false (F) for the following statements :

(i) The number of digits in a square number is even.

(ii) The square of a prime number is prime.

(iii) The sum of two square numbers is a square number.

(iv) The difference of two square numbers is a square number.

(v) The product of two square numbers is a square number.

(vi) No square number is negative.

(vii) There is no square number between 50 and 60.

(viii) There are fourteen square numbers upto 200.

1.4 Alternate Methods of Squaring Numbers

Finding the square of a given integer is a simple matter. You have only to multiply the given integer with itself. For large numbers, multiplication may prove to be laborious and time-consuming. In this section, we shall see how to find the square of two or three-digit numbers quickly without actual multiplication. The first method is based upon an old Indian method of multiplying two numbers, which we shall call *Column method*. The method for squaring a two digit number uses the identity :

$$(a + b)^2 = a^2 + 2ab + b^2$$

To square a two-digit number ab (where a is the tens digit and b is the units digit), we make three columns and write a^2, $2a \times b$ and b^2, respectively in these columns as follows (As an illustration, we take $ab = 86$) :

Column I	*Column* II	*Column* III
a^2 ($8^2 = 64$)	$2a \times b$ ($2 \times 8 \times 6 = 96$)	b^2 ($6 \times 6 = 36$)

We then go through the following steps :

Step 1 : Underline the units digit of b^2 (in Column III) and add the tens digit of b^2, if any, to $2a \times b$ in Column II.

I	II	III
a^2	$2a \times b$	b^2
64	96	$3\underline{6}$
	+ 3	
	99	

Step 2 : Underline the units digit in Column II and add the tens digit, if any, to a^2 in Column I.

a^2	$2a \times b$	b^2
64	96	$3\underline{6}$
+ 9	+ 3	
73	$9\underline{9}$	

Step 3 : Underline the number in Column I.

a^2	$2a \times b$	b^2
64	96	$3\underline{6}$
+ 9	+ 3	
$\underline{73}$	$9\underline{9}$	

$86^2 = 7396$

The underlined digits give the required square.

Example 1 : Find the squares of (i) 65 and (ii) 37.

Solution:

(i) 65×65

36	60	$2\underline{5}$
+ 6	+ 2	
$\underline{42}$	$6\underline{2}$	

$65^2 = 4225$

(ii) 37×37

9	42	$4\underline{9}$
+ 4	+ 4	
$\underline{13}$	$4\underline{6}$	

$37^2 = 1369$

As the number of digits increases, the Column method becomes difficult. In this case, we may use the following method which we shall call the *Diagonal method*. This again is an old Indian method of multiplying two numbers. However, we illustrate it here for squaring 25, 36 and 486.

First we form a square. Then we divide it into sub-squares, draw some diagonals and write the digits of the number to be squared as shown here :

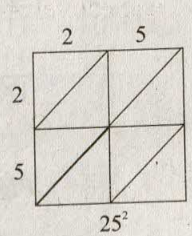


Fig. 1.1 (i)

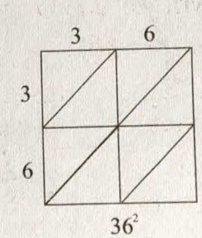


Fig. 1.2 (i)

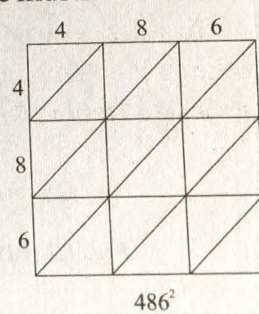


Fig. 1.3 (i)

Now multiply each digit on the left of the square with each digit on top of the column one-by-one. Write the product in the corresponding sub-square [Fig. 1.1(ii), Fig. 1.2 (ii) and Fig. 1.3 (ii)]. If the number so obtained is a single digit number, write it below the diagonal. If it is a two digit number, write the tens digit above the diagonal and the units digit below the diagonal.

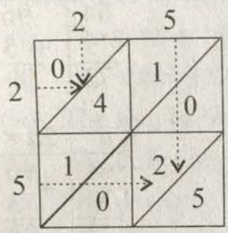


Fig. 1.1 **(ii)**

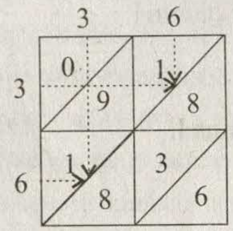


Fig. 1.2 **(ii)**

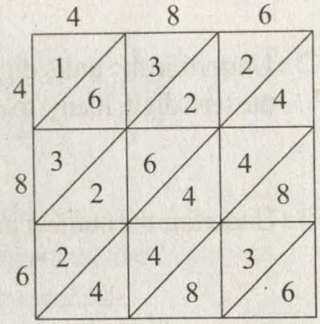
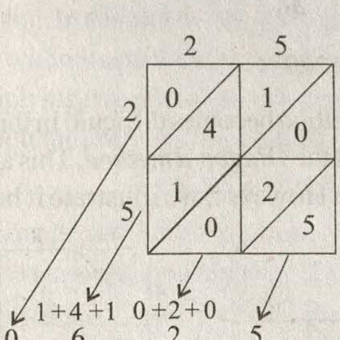


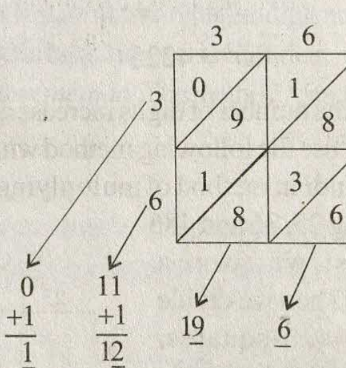
Fig. 1.3 **(ii)**

Starting below the lowest diagonal, sum the digits along the diagonals so obtained, underline the units digit of the sum and carry over the tens digit, if any, to the diagonal above. The units digits so underlined together with all the digits in the sum above the top-most diagonal give the square. The numbers in the empty places are taken as zero [Fig. 1.1(iii), Fig. 1.2 (iii) and Fig. 1.3 (iii)].



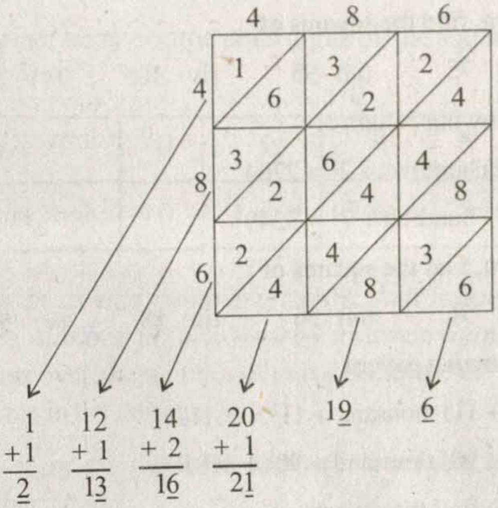
$25^2 = 625$

Fig. 1.1 **(iii)**



$36^2 = 1296$

Fig. 1.2 **(iii)**



$$486^2 = 236196$$

Fig. 1.3 **(iii)**

Remark : The Diagonal method can be applied to find the square of any number irrespective of the number of digits.

EXERCISE 1.2

1. Find the squares of the following numbers using the Column method. Verify the result by finding the square using the usual method.

(i) 25 (ii) 37 (iii) 54 (iv) 96 (v) 71

2. Find the squares of the following numbers using Diagonal method :

(i) 89 (ii) 276 (iii) 349 (iv) 293 (v) 161

3. Find the square of :

(i) 127 (ii) 235 (iii) 852 (iv) 251 (v) 501

4. Consider the following pattern :

$25^2 = 2 \times (2 + 1)$ hundred $+ 25 = 625$

$45^2 = 4 \times (4 + 1)$ hundred $+ 25 = 2025$

$115^2 = 11 \times (11 + 1)$ hundred $+ 25 = 13225$

Using this pattern, find the squares of :

(i) 35 (ii) 75 (iii) 95 (iv) 105 (v) 205

5. Consider the following pattern :

$$52^2 = (5^2 + 2) \text{ hundred} + 2^2 = 2704$$

$$57^2 = (5^2 + 7) \text{ hundred} + 7^2 = 3249$$

Using this pattern, find the squares of :

(i) 51 (ii) 54 (iii) 56 (iv) 58 (v) 59

6. Consider the following pattern :

$$511^2 = (250 + 11) \text{ thousand} + 11^2 = 261121$$

$$590^2 = (250 + 90) \text{ thousand} + 90^2 = 348100$$

Using this pattern find the squares of :

(i) 509 (ii) 515 (iii) 525 (iv) 580 (v) 534

7. Find the squares of following numbers using the identity :

$$(a + b)^2 = a^2 + 2ab + b^2$$

(i) 509 (ii) 211 (iii) 625

8. Find the squares of the following numbers using the identity :

$$(a - b)^2 = a^2 - 2ab + b^2$$

(i) 491 (ii) 189 (iii) 575

1.5 Square Roots

If $n = m^2$, then we call m a *square root* of n. Thus, a square root of 4 is 2, since $4 = 2^2$. Similarly, the square roots of 25 are 5 and -5, since $25 = 5^2$ and $25 = (-5)^2$. Likewise, square root of 49 is 7, square root of 121 is 11, etc. Therefore, if n is a perfect square, then its square root is an integer. If n is not a perfect square, then there is no integer m such that square root of n is m, i.e., it does not have an integral square root. Throughout this section, square stands for perfect square and square root means integral square root.

Based on the properties of square numbers discussed in Section 1.3, we have the following propositions about square roots :

I. *If the units digit of a number is* 2, 3, 7 *or* 8, *then it is not a perfect square and hence does not have a square root.*

II. *If a number has a square root, then its units digit must be* 0, 1, 4, 5, 6 *or* 9.

By Property II of square numbers, the units digits of the square and square root are related as below :

Units digit of square :	0	1	4	5	6	9
Units digit of square root :	0	1 or 9	2 or 8	5	4 or 6	3 or 7

III. *If a number ends in an odd number of zeros, then it does not have a square root. If a square number is followed by an even number of zeros, it has a square root*. Moreover, the number of zeros at the end of the square root is half the number of zeros in the number.

IV. *The square root of an even square number is even and the square root of an odd square number is odd.*

V. Note that $2^2 = 4$, $3^2 = 9$, $4^2 = 16$ and so on. Also, $(-2)^2 = (-2) \times (-2) = 4$, $(-3)\times(-3)=9$, $(-4)\times(-4)=16$ and so on. Clearly, the square of a number, whether positive or negative is always positive. In other words, *negative numbers are not perfect squares and, therefore, have no square root in the system of rational numbers*. (However, negative numbers do have square roots in some other number systems.)

As we saw above, $2^2=(-2)^2=4$. Therefore, both of 2 and -2 are square roots of 4. Similarly, other square numbers also have two square roots–one positive and the other negative. However, at this stage we consider only the positive square root. Further, symbolically, we write the positive square root of 4 as $\sqrt[2]{4}$ or simply as $\sqrt{4}$. The symbol '$\sqrt{\ }$' stands for the positive square root. Thus $\sqrt{4} = 2$ and $\sqrt{4} \neq -2$.

We shall now discuss some methods of finding square roots of perfect square numbers.

We observe that

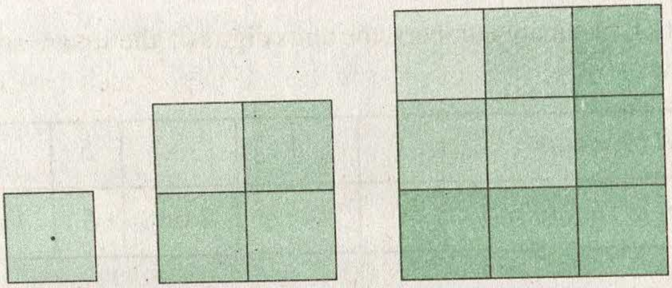
$$1 = 1^2$$

$$1 + 3 = 2^2 \quad \text{(Sum of the first } \textit{two} \text{ odd numbers} = 2^2\text{)}$$

$$1 + 3 + 5 = 3^2 \quad \text{(Sum of the first } \textit{three} \text{ odd numbers} = 3^2\text{)}$$

In general, the sum of first n odd numbers is n^2.

i.e., $1 + 3 + 5 + \ldots + (2n - 1) = n^2$



1     $1 + 3$ $1 + 3 + 5$

Fig. 1.4

We may use this result to find the square root of a perfect square as follows :

Take the number n whose square root is required. Subtract from n the odd numbers 1, 3, 5, . . . successively. If n is a perfect square, we will get zero at some stage. We stop at this point and declare the number of times we have performed subtraction, as the square root of n. For example, let us consider 49 $(= 7^2)$.

Now, (i) $49-1 = 48$, (ii) $48-3 = 45$, (iii) $45-5 = 40$, (iv) $40-7=33$, (v) $33 - 9 = 24$, (vi) $24 - 11 = 13$, (vii) $13 - 13 = 0$.

Here, the total number of subtractions is 7, therefore, $\sqrt{49} = 7$.

This is the simplest method of finding the square root of a perfect square. It works very well for small numbers. However, it is a long and time-consuming process for large numbers. We have other more efficient methods for extracting square roots.

1.6 Prime Factorisation Method for Square Roots

Consider the following prime factorisations :

$$6 = 2 \times 3 \qquad 6^2 = (2 \times 3) \times (2 \times 3) = \underline{2 \times 2} \times \underline{3 \times 3}$$

$$8 = 2 \times 2 \times 2 \qquad 8^2 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2}$$

$$12 = 2 \times 2 \times 3 \qquad 12^2 = (2 \times 2 \times 3) \times (2 \times 2 \times 3) = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3}$$

We observe that

(i) If p is a prime factor of n, then $p \times p$ is a factor of n^2.

(ii) If p is a prime and $p \times p$ is a factor of n^2, then p is a factor of n.

(iii) Prime factors of n^2 may be paired so that the two factors in every pair are equal.

Therefore, we have the following steps for finding the square root of a perfect square n :

(i) Write the prime factorisation of n. Pair the factors such that primes in each pair are equal.

(ii) Choose one prime from each pair and multiply all such primes.

(iii) The product obtained in (ii) is the square root of n.

Example 2 : Find the square root of 8100.

Solution : $8100 = \underline{2 \times 2} \times \underline{3 \times 3} \times \underline{3 \times 3} \times \underline{5 \times 5}$

$\therefore \quad \sqrt{8100} = 2 \times 3 \times 3 \times 5 = 90$

2	8100
2	4050
3	2025
3	675
3	225
3	75
5	25
	5

Example 3 : Is 2352 a perfect square? If not, find the smallest number by which 2352 must be multiplied so that the product is a perfect square. Find the square root of the new number.

Solution : $2352 = \underline{2 \times 2} \times \underline{2 \times 2} \times 3 \times \underline{7 \times 7}$

Here, we find that the prime 3 does not occur in a pair. Therefore, 2352 is not a perfect square.

If we multiply 2352 by 3, then

$2352 \times 3 = 7056 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3} \times \underline{7 \times 7}$

Now each prime occurs in a pair. Therefore, 2352×3, i.e., 7056 is a perfect square. Thus, the required smallest number is 3.

Also, $\sqrt{7056} = 2 \times 2 \times 3 \times 7 = 84$

2	2352
2	1176
2	588
2	294
3	147
7	49
	7

Example 4 : Find the smallest number by which 9408 must be divided so that the quotient is a perfect square. Find the square root of the quotient.

Solution : $9408 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2} \times 3 \times \underline{7 \times 7}$

If we divide 9408 by 3, then

$9408 \div 3 = 3136 = \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{2 \times 2} \times \underline{7 \times 7}$, which is a perfect square.

Therefore, the required smallest number is 3.

Hence, $\sqrt{3136} = 2 \times 2 \times 2 \times 7 = 56$

EXERCISE 1.3

1. Which of the following numbers are not perfect squares?
   (i) 81 (ii) 92 (iii) 121 (iv) 132
2. Check if the following numbers are perfect second powers :
   (i) 153 (ii) 257 (iii) 408 (iv) 441
3. Write the possible units digits of the square roots of the following numbers. Which of these numbers have odd square roots?
   (i) 9801 (ii) 99856 (iii) 998001 (iv) 657666025
4. Find the square roots of 121 and 169 by the method of repeated subtraction.
5. Find the square roots of the following numbers by the Prime Factorisation Method:
   (i) 729 (ii) 400 (iii) 1764 (iv) 4096
6. Write the prime factorisation of the following numbers and hence find their square roots :
   (i) 7744 (ii) 9604 (iii) 5929 (iv) 7056
7. Check if the following numbers are perfect second powers. If yes, find their square roots.
   (i) 1936 (ii) 8281
8. For each of the following numbers, find the smallest number with which it should be multiplied so as to get a perfect square. Also find the square root of the square number so obtained.
   (i) 180 (ii) 1458 (iii) 1200 (iv) 1008 (v) 2028
9. For each of the following numbers, find the smallest number by which it should be divided so as to get a perfect square. Also find the square root of the square number so obtained.
   (i) 180 (ii) 3645 (iii) 2800 (iv) 45056
10. The students of Class VIII of a school donated Rs 2401 for Prime Minister's National Relief Fund. Each student donated as many rupees as the number of students in the class. Find the number of students in the class.
11. A PT teacher wants to arrange maximum possible number of 6000 students in a field such that the number of rows is equal to the number of columns. Find the number of rows if 71 students were left out after the arrangement.

1.7 Square Roots by Division Method

The method of finding square roots by the prime factorisation method is efficient only if the number has small prime factors. Sometimes it is difficult and time consuming to obtain

prime factors of a given number. To overcome this difficulty, we use an alternative method called the *division method*. For this we need to know the number of digits in the square root of a perfect square.

We know that

$$1^2 = 1,\ 9^2 = 81 \text{ and } 10^2 = 100$$

i.e., the square of a single digit number is at the most a 2-digit number. Since the smallest 3-digit number is 100 and its square root 10 is a 2-digit number, the square root of a one or two digit (perfect square) number is a single digit number.

Likewise $10^2 = 100$, $99^2 = 9801$ and $100^2 = 10000$. This shows that if a perfect square number has three or four digits, then its square root will be a two-digit number. Similarly, if a perfect square number has five or six digits, its square root will contain three digits, and so on.

A quicker way to determine the number of digits in the square root of a square number is to place a bar over every pair of digits starting from the units digit. If the number of digits in n is odd, then the left–most single digit too has a bar. The number of bars is the number of digits in $\sqrt{n}$. For example, if $n = 256$, then $\sqrt{n}$ has two digits, because there are two bars ($\overline{2}\ \ \overline{56}$) here. The square root of 783225 has three digits because there are three bars $\left(\overline{78}\ \ \overline{32}\ \ \overline{25}\right)$ here.

Let us illustrate the division method by taking the number 531441.

Step 1 : Place a bar over every pair of digits starting from the units digit. $\overline{53}\,\overline{14}\,\overline{41}$

Step 2 : Find the largest number whose square is less than or equal to the number under the left–most bar ($7^2 < 53 < 8^2$). Take this number as the divisor and the number under the left-most bar as the dividend. Divide and get the remainder. (Note that the quotient in this particular step is the same as the divisor.)

$$\begin{array}{r|l} & \mathbf{7} \\ \hline 7 & \mathbf{\overline{53}}\ \overline{14}\ \overline{41} \\ & \underline{\mathbf{49}} \\ & \mathbf{4} \end{array}$$

Step 3 : Bring down the number under the next bar to the right of the remainder. This is the new dividend.

$$\begin{array}{r|l} & 7 \\ \hline 7 & \overline{53}\ \overline{14}\ \overline{41} \\ & \underline{49} \\ & \mathbf{4\ 14} \end{array}$$

Step 4 : Double the quotient and enter it with a blank on the right for the next digit of the next possible divisor.

$$\begin{array}{r|l} & 7 \\ \hline 7 & \overline{53}\ \overline{14}\ \overline{41} \\ & \underline{49} \\ \mathbf{14-} & \mathbf{4\ 14} \end{array}$$

Step 5 : Guess a largest possible digit to fill the blank and also to become the new digit in the quotient.

$(142 \times 2 = 284 < 414,\ 143 \times 3 = 429 > 414)$

Get the remainder.

$$\begin{array}{r|l} & \quad 72 \\ \hline 7 & \overline{53}\ \overline{14}\ \overline{41} \\ & 49 \\ \cline{2-2} \mathbf{142} & \ \mathbf{414} \\ & \ \mathbf{284} \\ \cline{2-2} & \ \mathbf{130} \end{array}$$

Step 6 : Bring down the number under the next bar to the right of the new remainder.

$$\begin{array}{r|l} & \quad 72 \\ \hline 7 & \overline{53}\ \overline{14}\ \overline{41} \\ & 49 \\ \cline{2-2} 142 & \ 414 \\ & \ 284 \\ \cline{2-2} & \ \mathbf{13041} \end{array}$$

Step 7 : Repeat Steps 4, 5 and 6 till all bars have been considered. The final quotient is the square root.

Thus, $\sqrt{531441} = 729$

$$\begin{array}{r|l} & \quad 729 \\ \hline 7 & \overline{53}\ \overline{14}\ \overline{41} \\ & 49 \\ \cline{2-2} 142 & \ 414 \\ & \ 284 \\ \cline{2-2} \mathbf{1449} & \ \mathbf{13041} \\ & \ \mathbf{13041} \\ \cline{2-2} & \qquad \mathbf{0} \end{array}$$

Remark : To determine 2 after 14 in Step 5, we try $41 \div 14$. Similarly, to determine 9 after 144 in Step 7, we try $1304 \div 144$ or $130 \div 14$. Here 9 is the units digit of the square root. It can also be determined by the units digit of the square number. Here the square number is 531441 and its units digit is 1. Therefore, the units digit of its square root is 1 or 9. Since 1 can be easily ruled out, the required digit is 9.

Example 5 : Find the square root of 363609.

Solution :

$$\begin{array}{r|l} & \quad 603 \\ \hline 6 & \overline{36}\ \overline{36}\ \overline{09} \\ & 36 \\ \cline{2-2} 120 & \ 0\ 36 \\ & \ 0\ 00 \\ \cline{2-2} 1203 & \quad 3609 \\ & \quad 3609 \\ \cline{2-2} & \qquad 0 \end{array}$$

$\therefore \qquad \sqrt{363609} = 603$

Remark: Here we get 0 after 12 by trying $03 \div 12$. We guess 3 after 120 because the units digit in the square number is 9 (or because $360 \div 120 = 3$).

Though the above method is more efficient with larger numbers, it can be used to find square root of smaller three or four digit numbers also.

Example 6 : Find the square roots of (i) 529, (ii) 1296

Solution : (i)

$$\begin{array}{r|l} & 23 \\ \hline 2 & \overline{5}\ \overline{29} \\ & 4 \\ \hline 43 & 129 \\ & 129 \\ \hline & 0 \end{array}$$

(ii)

$$\begin{array}{r|l} & 36 \\ \hline 3 & \overline{12}\ \overline{96} \\ & 9 \\ \hline 66 & 396 \\ & 396 \\ \hline & 0 \end{array}$$

$\therefore \qquad \sqrt{529} = 23 \qquad\qquad \therefore \qquad \sqrt{1296} = 36$

Example 7 : Find the least number that must be subtracted from 893304 so as to get a perfect square.

Solution :

$$\begin{array}{r|l} & 945 \\ \hline 9 & \overline{89}\ \overline{33}\ \overline{04} \\ & 81 \\ \hline 184 & 833 \\ & 736 \\ \hline 1885 & 9704 \\ & 9425 \\ \hline & 279 \end{array}$$

Thus, if we subtract 279 from 893304, we get a perfect square number (whose square root is 945).

Example 8 : Find the least number which must be added to 893304 to obtain a perfect square.

Solution : Working as above, we see that 893304 is greater than 945^2. The next perfect square is 946^2, i.e., 894916.

Hence, the number to be added is 894916 – 893304, i.e., 1612.

For square numbers upto four digits, we may find the square root without doing the division process as follows :

Step 1 : Find the largest number whose square is less than or equal to the number under the left – most bar. This is the tens digit of the square root.

Step 2 : Guess the units digit by the Table given in Section 1.5. Units digit is 1 or 9.

Step 3 : Choose the correct digit by squaring one possible square root and comparing with the given number.

Illustration : $\sqrt{\overline{98}\ \overline{01}} = ?$

$\because$ $9^2 = 81$ is the largest square number ≤ 98

$\therefore$ $\sqrt{\overline{98}\ \overline{01}} = 9\square$

$91^2 = 8281 \neq 9801$

$\therefore$ $\sqrt{9801} = 99$

Example 9 : Find the square roots of :

(i) 256 (ii) 6561

Solution : (i) Putting bars, we have : $\overline{2}\ \ \overline{56}$. Therefore, the tens digit is 1.

Also, possible units digits of the square root are 4 and 6. So the root is 14 or 16.

Further, $14^2 = 196 \neq 256$

$\therefore$ $\sqrt{256} = 16$

(ii) $\sqrt{6561} = 81$ or 89

Also, 6561 is closer to 6400 ($= 80^2$) than to 8100 ($= 90^2$).

$\therefore$ $\sqrt{6561}$ must be closer to 80 than to 90.

Thus, $\sqrt{6561} = 81$

EXERCISE 1.4

1. Find the number of digits in the square roots of the following numbers :

   (i) 64 (ii) 144 (iii) 4489 (iv) 27225 (v) 390625

2. Put a dot on the units digit of a number. Now put a dot over every alternate digit. The number of dots gives the number of digits in the square root of the given number. Find the number of digits in the square root of :

   (i) 1234321 (ii) 21224449 (iii) 3915380329

3. Using the division method, find the square roots of the following numbers :
(i) 44100 (ii) 27225 (iii) 54756 (iv) 49284 (v) 99856

4. Using the division method, find the square roots of the following :
(i) 390625 (ii) 119025 (iii) 193600

5. Find the square roots of the following numbers by finding their units and tens digits :
(i) 2304 (ii) 4489 (iii) 3481 (iv) 529

6. Find the square roots of the following numbers :
(i) 1444 (ii) 1849 (iii) 5776 (iv) 7921

7. Find the least numbers which must be subtracted from the following numbers so as to leave a perfect square :
(i) 2361 (ii) 4931 (iii) 18265 (iv) 390700

8. For each of the numbers in Question 7, find the smallest number that must be added to get a perfect square.

1.8 Square Root of a Rational Number

Consider the perfect squares 25 and 36. Now

$$\sqrt{25 \times 36} = \sqrt{5^2 \times 6^2}$$

$= \sqrt{(5 \times 6)^2}$, since we know that for integers a and b

$(a \times b)^2 = a^2 \times b^2$

$= 5 \times 6$

$= \sqrt{25} \times \sqrt{36}$

In fact, we have :

Rule 1 : For perfect squares m and n, $\sqrt{m \times n} = \sqrt{m} \times \sqrt{n}$

This Rule is quite helpful in determining the square roots of large numbers.

Example 10 : Find the square root of 38416.

Solution : $38416 = 2 \times 2 \times 2 \times 2 \times 7 \times 7 \times 7 \times 7$

$= 2^4 \times 7^4$

$\therefore \quad \sqrt{38416} = \sqrt{2^4 \times 7^4}$

$= \sqrt{2^4} \times \sqrt{7^4}$, by using Rule 1

$= 2^2 \times 7^2$

$= 196$

Example 11 : Find $\sqrt{\frac{25}{36}}$ and $\frac{\sqrt{25}}{\sqrt{36}}$. Are they equal ?

Solution : $\sqrt{\frac{25}{36}} = \sqrt{\frac{5^2}{6^2}}$

$= \sqrt{\left(\frac{5}{6}\right)^2}$, since $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}, b \neq 0$

$= \frac{5}{6}$

Also, $\frac{\sqrt{25}}{\sqrt{36}} = \frac{5}{6}$

Thus, $\sqrt{\frac{25}{36}} = \frac{5}{6} = \frac{\sqrt{25}}{\sqrt{36}}$

In the above example, we have illustrated the following rule :

Rule 2 : If m and n are perfect squares (and $n \neq 0$), then

$$\sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$$

Example 12 : Find the square root of $\frac{225}{3136}$.

Solution : $\sqrt{225} = \sqrt{3 \times 3 \times 5 \times 5}$

$= 3 \times 5 = 15$, and

$\sqrt{3136} = \sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 7 \times 7}$

$= 2 \times 2 \times 2 \times 7$

$= 56$

$\therefore \sqrt{\frac{225}{3136}} = \frac{\sqrt{225}}{\sqrt{3136}}$ (By Rule 2)

$= \frac{15}{56}$

Example 13 : Find the square roots of (i) $4\frac{29}{49}$ and (ii) 0.0196.

Solution : (i) $\sqrt{4\frac{29}{49}} = \sqrt{\frac{225}{49}}$

$$= \frac{\sqrt{225}}{\sqrt{49}} \qquad \text{(By Rule 2)}$$

$$= \frac{15}{7} = 2\frac{1}{7}$$

(ii) $\sqrt{0.0196} = \sqrt{\frac{196}{10000}}$

$$= \frac{\sqrt{196}}{\sqrt{10000}} \qquad \text{(By Rule 2)}$$

$$= \frac{\sqrt{2 \times 2 \times 7 \times 7}}{\sqrt{100 \times 100}}$$

$$= \frac{2 \times 7}{100} = 0.14$$

Example 14 : Find the square root of $21\frac{2797}{3364}$.

Solution : $\sqrt{21\frac{2797}{3364}} = \sqrt{\frac{73441}{3364}} = \frac{\sqrt{73441}}{\sqrt{3364}}$ (By Rule 2)

Now, $\sqrt{73441} = 271$, and

$\sqrt{3364} = 58$

$\therefore \quad \sqrt{21\frac{2797}{3364}} = \frac{271}{58}$

$$= 4\frac{39}{58}$$

$$\begin{array}{r|l} & \;\;271 \\ \hline 2 & \overline{7}\;\overline{34}\;\overline{41} \\ & 4 \\ \hline 47 & \;334 \\ & \;329 \\ \hline 541 & \quad 541 \\ & \quad 541 \\ \hline & \qquad 0 \end{array} \qquad \begin{array}{r|l} & \;\;58 \\ \hline 5 & \overline{33}\;\overline{64} \\ & 25 \\ \hline 108 & \;864 \\ & \;864 \\ \hline & \quad 0 \end{array}$$

Example 15 : Find the square root of 37.0881.

Solution : We may convert 37.0881 into a rational number and then find the square root either by factorisation or by the division method.

Thus, $\sqrt{37.0881} = \sqrt{\dfrac{370881}{10000}}$

But $\sqrt{370881} = 609$

$\sqrt{10000} = 100$

$\therefore$ $\sqrt{37.0881} = \dfrac{609}{100}$

$= 6.09$

$$\begin{array}{r|l} & 609 \\ \hline 6 & \overline{37}\;\overline{08}\;\overline{81} \\ & 36 \\ \hline 120 & 108 \\ & 0 \\ \hline 1209 & 10881 \\ & 10881 \\ \hline & 0 \end{array}$$

1.9 Square Root of a (Perfect Square) Decimal Number

We may find the square root of a decimal number without converting it into a rational number. We do it as follows :

1. Place bars on the integral part of the number in the usual manner.
2. Place bars on the decimal part on every pair of digits beginning with the first decimal place.
3. Start finding the square root by the division process as usual.
4. Place decimal point in the quotient as soon as the integral part is exhausted.
5. Stop when the remainder becomes zero. The quotient at this stage is the square root.

Example 16 : Find the square root of 37.0881 by the division method.

Solution :

$$\begin{array}{r|l} & 6.09 \\ \hline 6 & \overline{37}.\;\overline{08}\;\overline{81} \\ & 36 \\ \hline 120 & 108 \\ & 0 \\ \hline 1209 & 10881 \\ & 10881 \\ \hline & 0 \end{array}$$

$\therefore$ $\sqrt{37.0881} = 6.09$

Example 17 : Find the square root of 0.000529.

Solution :

$$\begin{array}{r|l} & \quad 0.023 \\ \hline 2 & \overline{0}.\ \overline{00}\ \overline{05}\ \overline{29} \\ & \qquad\ \ 04 \\ \cline{2-2} 43 & \qquad\ 129 \\ & \qquad\ 129 \\ \cline{2-2} & \qquad\quad 0 \end{array}$$

$\therefore \qquad \sqrt{0.000529} = 0.023$

Remark : In this example, the integral part is zero. Therefore, the integral part of the square root is also zero. Again, the first pair after the decimal point is 00. Therefore, the first digit after the decimal point in the square root is zero. The rest of the procedure is as usual.

1.10 Approximate Square Root by the Division Method

We have observed that it is not possible to find integers p and q such that $p^2 = 2q^2$ or $2 = \frac{p^2}{q^2}$. In other words, there is no rational number $\frac{p}{q}$ such that $\sqrt{2} = \frac{p}{q}$. It means we cannot calculate $\sqrt{2}$ exactly. We can only find a rational number which is approximately equal to $\sqrt{2}$. In such a situation, where it is not possible to calculate the exact value of an expression, we try to find an approximate value. In the present situation, we try to find a rational or decimal number approximately equal to $\sqrt{2}$.

Since $\qquad 1^2 < 2 < 2^2,$

therefore, $\qquad 1 < \sqrt{2} < 2$

Thus, 1 may be taken as a rational number which is approximately equal to $\sqrt{2}$. To find a rational number closer to $\sqrt{2}$, we square 1.1, 1.2, . . . , 1.9, all lying between 1 and 2, and compare the squares with 2.

$$1.1^2 = 1.21 < 2$$
$$1.2^2 = 1.44 < 2$$
$$1.3^2 = 1.69 < 2$$
$$1.4^2 = 1.96 < 2$$
$$1.5^2 = 2.25 > 2$$

Thus, $\qquad (1.4)^2 < 2 < (1.5)^2$

i.e., $1.4 < \sqrt{2} < 1.5$

We may now take 1.4 as a decimal number which is approximately equal to $\sqrt{2}$. It is a better approximation than 1. To find a still better approximation, we square numbers 1.41, 1.42, . . . etc. and compare these squares with 2. By doing so, we find that $1.41 < \sqrt{2} < 1.42$. Continuing further, we may get

$$1.414 < \sqrt{2} < 1.415$$

Continuing this way, we can find rational numbers which are nearer and nearer to $\sqrt{2}$, upto a certain number of decimal places. A convenient way of finding such a number is to use the division method. We may add a suitable number of zeros to the right of the decimal part of the number before applying this method. We illustrate it using suitable examples.

Example 18 : Find the square root of 2 correct to two places of decimal.

Solution : To find the number which is equal to $\sqrt{2}$ correct to two places of decimal, we find a number which is approximately equal to $\sqrt{2}$ and has three places of decimal. For this purpose, we append three pairs of zeros, i.e., six zeros to the right of the decimal point.

$$\begin{array}{r|l} & 1.414 \\ \hline 1 & \overline{2}.\,\overline{00}\,\overline{00}\,\overline{00} \\ & 1 \\ \hline 24 & 100 \\ & 96 \\ \hline 281 & 400 \\ & 281 \\ \hline 2824 & 11900 \\ & 11296 \\ \hline & 604 \end{array}$$

Hence, $\sqrt{2} = 1.414$ upto three places of decimal

$= 1.41$ correct upto two places of decimal

Thus, the required square root of 2 is 1.41.

Example 19 : Find the square root of 2.9 correct upto two places of decimal.

Solution : First we find a decimal number approximately equal to $\sqrt{2.9}$ which has three places of decimal. To do so, we append five zeros to make three pairs after the decimal point.

$$\begin{array}{r|l} & 1.702 \\ \hline 1 & \overline{2}.\,\overline{90}\,\overline{00}\,\overline{00} \\ & 1 \\ \hline 27 & 190 \\ & 189 \\ \hline 3402 & 10000 \\ & 6804 \\ \hline & 3196 \end{array}$$

Hence, $\sqrt{2.9} = 1.702$ upto three places of decimal

$= 1.70$ correct upto two places of decimal

Thus, the required square root is 1.70.

Example 20 : Find the square root of $11\frac{2}{3}$ correct to two places of decimal.

Solution : $11\frac{2}{3} = 11.666666\ldots$

$= 11.666667$ correct upto six decimal places

$\therefore \quad \sqrt{11\frac{2}{3}} = \sqrt{11.666667}$

We find the square root as follows :

$$
\begin{array}{r|l}
 & \quad 3.415 \\
\hline
3 & \overline{11}.\,\overline{66}\,\overline{66}\,\overline{67} \\
 & 9 \\
\cline{2-2}
64 & 266 \\
 & 256 \\
\cline{2-2}
681 & 1066 \\
 & 681 \\
\cline{2-2}
6825 & 38567 \\
 & 34125 \\
\cline{2-2}
 & 4442
\end{array}
$$

$\therefore \quad \sqrt{11\frac{2}{3}} = 3.415$ upto three decimal places

$= 3.42$ correct upto two decimal places.

Remark: We may also find $\sqrt{11\frac{2}{3}}$ i.e., $\sqrt{\frac{35}{3}}$ by making the denominator free from radical sign.

$$\sqrt{\frac{35}{3}} = \frac{\sqrt{35}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{105}}{3}$$

By the division method,

$\sqrt{105} = 10.246$ upto three decimal places

$\therefore \quad \sqrt{\frac{35}{3}} = \frac{\sqrt{105}}{3} = \frac{10.246}{3}$ upto three decimal places

$= 3.415$ upto three decimal places

$= 3.42$ correct upto two decimal places.

EXERCISE 1.5

1. Find the square roots of the following rational numbers :

 (i) $\frac{361}{625}$ (ii) $\frac{2116}{15129}$

2. Find the square root of

 (i) $\frac{16641}{4489}$ (ii) $\frac{110889}{308025}$

3. Find the square roots of the following mixed numbers :

 (i) $21\frac{51}{169}$ (ii) $10\frac{151}{225}$

4. Find the square root of

 (i) $23\frac{394}{729}$ (ii) $56\frac{569}{1225}$

5. Find the square roots of the following decimal numbers :

 (i) 7.29 (ii) 16.81 (iii) 9.3025 (iv) 84.8241

6. Find the square root of

 (i) 0.008281 (ii) 0.053361

7. Find the square roots of the following numbers correct to two places of decimal :

 (i) 1.7 (ii) 23.1 (iii) 5 (iv) 20 (v) 0.1

8. Calculate the square roots of the following numbers upto two places of decimal :

 (i) 0.016 (ii) 0.9 (iii) 7 (iv) $\frac{7}{8}$ (v) $2\frac{1}{12}$

9. Find the square roots of the following numbers correct to three places of decimal :

 (i) 0.00064 (ii) $\frac{5}{12}$ (iii) 2.006 (iv) 1.1

10. Write true (T) or false (F) for the following statements :

 (i) $\sqrt{0.9} = 0.3$

 (ii) If a is a natural number, then $\sqrt{a}$ is a rational number

 (iii) If a is negative, then a^2 is also negative.

 (iv) If p and q are perfect squares, then $\sqrt{\frac{p}{q}}$ is a rational number.

 (v) The square root of a prime number is not a rational number.

Things to Remember

1. A number n is a perfect square, if $n = m^2$ for some integer m.
2. A perfect square number is never negative.
3. A square number never ends in 2, 3, 7 or 8.
4. The number of zeros at the end of a perfect square is even.
5. The square of an even (odd) number is even (odd).
6. A perfect square number leaves a remainder 0 or 1 on division by 3.
7. There are no natural numbers p and q such that $p^2 = 2q^2$.
8. A number m is a square root of n, if $n = m \times m = m^2$. Positive square root of n is written as $\sqrt{n}$.
9. If p and q are perfect squares ($q \neq 0$), then

   (i) $\sqrt{p \times q} = \sqrt{p} \times \sqrt{q}$ (ii) $\sqrt{\frac{p}{q}} = \frac{\sqrt{p}}{\sqrt{q}}$
10. The square root of a perfect square number can be obtained by finding the prime factorisation of the square number.
11. The square root of a perfect square may also be obtained by the division method.
12. The pairing of numbers in the division method starts from the decimal point. For the integral part, it goes from right to left and for the decimal part, it goes from left to right.
13. If a positive number is not a perfect square, then an approximate value of its square root may be obtained by the division method.
14. If p and q are not perfect squares, then to find $\sqrt{\frac{p}{q}}$, we may express $\frac{p}{q}$ as a decimal number and then use the division method.
15. We may also find $\sqrt{\frac{p}{q}}$ by making the denominator free from radical sign.
16. If n is not a perfect square, then $\sqrt{n}$ is not a rational number.

CHAPTER

2

CUBES AND CUBE ROOTS

2.1 Introduction

Like squares and square roots in Chapter 1, we shall study cubes and cube roots in this Chapter. We first discuss some properties of perfect cubes. We will then discuss patterns of perfect cubes which help us in finding cube roots of small perfect cube numbers. Based on the relationship between units digits of numbers and units digits of their cubes, we discuss a method of finding the cube root of a perfect cube. This method applies to numbers having at the most six digits. The advantage of this method is that there are very few and simple calculations. We then discuss prime factorisation method of finding the cube roots. Like square roots, we do have a division method to find the cube root of a number. However, we do not discuss this method here, as it is a little difficult and beyond the scope of this book.

2.2 Cube of a Number and Perfect Cube Numbers

We know that if x is a non-zero number, then $x \times x \times x$, written as x^3, is called the *cube of x* or simply *x cubed*. Thus, 8 $(= 2 \times 2 \times 2)$ is the cube of 2 or 2 cubed. Similarly, 27 $(= 3 \times 3 \times 3)$ is the cube of 3 or 3 cubed. Table 2.1 gives the cubes of the single digits 1 to 9.

Table 2.1 : *Cubes of the digits* 1 *to* 9

x	1	2	3	4	5	6	7	8	9
x^3	1	8	27	64	125	216	343	512	729

Each of the numbers 1, 8, 27, . . ., 729 is the cube of some integer or the other. Such numbers are called perfect cubes (or perfect third powers).

A number n is a perfect cube if there is an integer m such that $n = m \times m \times m$.

Perfect cubes grow very fast. As m grows from 1 to 9, the perfect cube m^3 grows from 1 to 729. Perfect cubes are fairly scattered. Upto 100, there are only four perfect cubes. Upto 1000, there are only ten perfect cubes. (Are you trying to check this fact? Note that $10^3 = 1000$.)

How do we examine whether a given number is a perfect cube?

If a prime p divides m, then $p \times p \times p$ will divide $m \times m \times m$, i.e., m^3. Therefore, *if a prime p divides a perfect cube, then p^3 also divides this perfect cube.*

In other words, in the prime factorisation of a perfect cube, every prime occurs three times or a multiple of three times. For example,

$$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \qquad (= 2^6)$$

$$27000 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \qquad (= 2^3 \times 3^3 \times 5^3)$$

Thus, to check whether a number is a perfect cube, we find its prime factorisation and group together triples of the same prime factors. If no factor is left out, the number is a perfect cube. However, if some factor is left as a single factor or a double factor, then the number is not a perfect cube.

Example 1 : Examine if (i) 392 and (ii) 106480 are perfect cubes.

Solution : (i) $392 = \underline{2 \times 2 \times 2} \times 7 \times 7$

Here, 7 does not appear in a group of three. Hence, 392 is not a perfect cube.

(ii) $106480 = \underline{2 \times 2 \times 2} \times 2 \times 5 \times \underline{11 \times 11 \times 11}$

Here, one prime factor 2 and one prime factor 5 do not appear in a group of three. Therefore, 106480 is not a perfect cube.

Example 2 : Examine if 53240 is a perfect cube. If not, find the smallest number by which it must be multiplied so that the product is a perfect cube. Find also the smallest number by which it must be divided so that the quotient is a perfect cube.

Solution : $53240 = \underline{2 \times 2 \times 2} \times 5 \times \underline{11 \times 11 \times 11}$

Here, the prime 5 does not appear in a group of three. Hence, 53240 is not a perfect cube. Further, 5 appears only once. If we multiply the number by 5×5, then in the product, 5 will also appear in a group of three and the product will be a perfect cube. Thus, the required smallest number by which the given number should be multiplied is 5×5, i.e., 25.

Finally, if we divide the given number 53240 by 5, the resulting number has prime factors in a group of three. In fact, $53240 \div 5 = 10648 = 2 \times 2 \times 2 \times 11 \times 11 \times 11$. Hence, the required number in this case is 5.

Remarks 1 : From Table 2.1, we observe that cubes of the digits 1, 4, 5, 6 and 9 are numbers ending in the same digits 1, 4, 5, 6 and 9, respectively. However, 2 and 8 make a pair in the sense that the cube of 2 ends in 8 and the cube of 8 ends in 2. Numbers 3 and 7 make another pair of this kind, i.e., the cube of either ends in the other ($3^3 = 2\underline{7}, 7^3 = 34\underline{3}$). Further, $10^3 = 1000$ shows that if a number ends in a zero, its cube will end in three zeros. These observations will help us in finding the cube roots of perfect cubes.

2. If a number is negative, then its cube is also negative. For example,

$$(-1)^3 = (-1) \times (-1) \times (-1) = -1 = -1^3$$

$$(-2)^3 = (-2) \times (-2) \times (-2) = -8 = -2^3$$

$$(-5)^3 = -125 = -5^3, \quad (-m)^3 = -m^3$$

This shows that negative numbers may also be perfect cubes. This is in contrast to perfect squares, which are *never* negative.

2.3 Finding the Cube of a Two-digit Positive Number (Alternative Method)

The cube of a number can be obtained by multiplying the number with itself three times. To find x^3, we may first find x^2 and then $x^2 \times x$. Here we shall discuss an alternative method of finding x^3, when x is a two digit number.

Let $x = ab$, where a is the tens digit and b is the units digit. Recall that for $(ab)^2$, we formed three columns $a^2 \mid 2a \times b \mid b^2$. Like in the method of finding x^2, here too we form columns. We use the Identity

$$(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3.$$

(You will learn this Identity in Chapter 6.) For $(ab)^3$ we form four columns

$$a^3 \mid 3a^2 \times b \mid 3a \times b^2 \mid b^3$$

Rest of the procedure is as before, i.e., we retain the units digit after addition and take the remaining digits to the next column for addition. We shall illustrate the method by means of examples.

Example 3 : Find 42^3 using alternative method.

Solution : Here, $a = 4, b = 2$. Thus, the four columns are :

a^3	$3a^2 \times b$	$3a \times b^2$	b^3
4^3	$3 \times 4^2 \times 2$	$3 \times 4 \times 2^2$	2^3
$= 6\underline{4}$	$= 9\underline{6}$	$= 4\underline{8}$	$= \underline{8}$
$+ 10$	$+ 4$		
74	$10\underline{0}$		
74	0	8	8

$\therefore \quad 42^3 = 74088$

Example 4 : Find 87^3 by alternative method.

Solution : Here, $a = 8$, $b = 7$. In a situation where a and b are not small, it may not be possible to calculate $3a^2 \times b$ and $3a \times b^2$ quickly. In such a situation, we simplify the working as follows :

a^2	a^2	b^2	b^2
a	$3b$	$3a$	b
a^3	$3a^2 \times b$	$3a \times b^2$	b^3

i.e.,

64	64	49	49
× 8	× 21	× 24	× 7
512	1344	1176	343
+146	+ 121	+ 34	
658	1465	1210	
658	5	0	3

$\therefore \quad 87^3 = 658503$

Example 5 : Find the cubes of the following numbers using alternative method :

(i) 27 (ii) 45 (iii) 81

Solution : (i) For 27^3, we have

4	4	49	49
2	21	6	7
8	84	294	343
+11	+32	+ 34	
19	116	328	
19	6	8	3

$\therefore \quad 27^3 = 19683$

(ii) For 45^3, we have

16	16	25	25
4	15	12	5
64	240	300	125
+27	+ 31	+ 12	
91	271	312	
91	1	2	5

$\therefore \quad 45^3 = 91125$

(iii) For 81^3, we have

64	64	1	1
8	3	24	1
512	192	24	1
+19	+2		
531	194		
531	4	4	1

$\therefore \quad 81^3 = 531441$

EXERCISE 2.1

1. Write the units digit of the cube of each of the following numbers :

31, 109, 388, 833, 4276, 5922, 77774, 44447, 125125125

2. Find the cubes of the following numbers:

(i) 35 (ii) 56 (iii) 72 (iv) 402 (v) 650 (vi) 819

3. Find the cubes of the following numbers using alternative method :

(i) 35 (ii) 56 (iii) 72.

4. Which of the following numbers are not perfect cubes?

(i) 64 (ii) 216 (iii) 243 (iv) 1728

5. For each of the non-perfect cubes in Question 4, find the smallest number by which it must be multiplied so that the product is a perfect cube.

6. For each of the non-perfect cubes in Question 4, find the smallest number by which it must be divided so that the quotient is a perfect cube.

7. By taking three different values of n, verify the truth of the following statements :

(i) If n is even, then n^3 is also even.

(ii) If n is odd, then n^3 is also odd.

(iii) If n leaves remainder 1 when divided by 3, then n^3 also leaves 1 as remainder when divided by 3.

(iv) If a natural number n is of the form $3p + 2$, then n^3 is also a number of the same type.

8. Write true (T) or false (F) for the following statements :

(i) 392 is a perfect cube.

(ii) 8640 is not a perfect cube.

(iii) No perfect cube can end with exactly two zeros.

(iv) There is no perfect cube which ends in 4.

(v) For an integer a, a^3 is always greater than a^2.

(vi) If a and b are integers such that $a^2 > b^2$, then $a^3 > b^3$.

(vii) If a divides b, then a^3 divides b^3.

(viii) If a^2 ends in 9, then a^3 ends in 7.

(ix) If a^2 ends in 5, then a^3 ends in 25.

(x) If a^2 ends in an even number of zeros, then a^3 ends in an odd number of zeros.

2.4 Cube Roots

If n is a perfect cube, then for some integer m, $n = m^3$. Here, the number m is called the *cube root* of n. Thus, a number m is a cube root of a number n, if $m^3 = n$. For example :

2 is a cube root of 8, because $2^3 = 8$.

5 is a cube root of 125, because $5^3 = 125$.

11 is a cube root of 1331, because $11^3 = 1331$.

If m is a cube root of n, we write $m = \sqrt[3]{n}$.

Thus, $2 = \sqrt[3]{8}$, $5 = \sqrt[3]{125}$ and $11 = \sqrt[3]{1331}$.

In Tables 2.2 and 2.3, we list respectively all the perfect cubes upto 1000 and their cube roots.

Table 2.2

m	m^3
1	1
2	8
3	27
4	64
5	125
6	216
7	343
8	512
9	729
10	1000

Table 2.3

n	$\sqrt[3]{n}$
1	1
8	2
27	3
64	4
125	5
216	6
343	7
512	8
729	9
1000	10

Remark : We use the symbol '$\sqrt[3]{\ }$' to denote cube root in the manner we used '$\sqrt[2]{\ }$' to denote square root. It is a mere convenience that we omit 2 and use '$\sqrt{\ }$' to denote square root. Strictly speaking, we should use the symbol '$\sqrt[2]{\ }$' for square root. However, we never omit 3 while writing '$\sqrt[3]{\ }$' for cube roots.

We now discuss some methods of finding cube roots of perfect cubes.

2.5 Cube Root Through a Pattern

Like squares of natural numbers, cubes too have some interesting patterns:

$$2^3 = 8 \quad \therefore \quad 2^3 - 1^3 = 7 = 1 + 1 \times 6 = 1 + 2 \times 1 \times 3$$

$$3^3 = 27 \quad \therefore \quad 3^3 - 2^3 = 19 = 1 + 1 \times 6 + 2 \times 6 = 1 + 3 \times 2 \times 3$$

$$4^3 = 64 \quad \therefore \quad 4^3 - 3^3 = 37 = 1 + 1 \times 6 + 2 \times 6 + 3 \times 6 = 1 + 4 \times 3 \times 3$$

.

.

.

$$9^3 = 729 \quad \therefore \quad 9^3 - 8^3 = 217 = 1 + 1 \times 6 + 2 \times 6 + \ldots + 8 \times 6$$
$$= 1 + 9 \times 8 \times 3$$

Also,

$$1 = 1^3$$
$$1 + 7 = 2^3$$
$$1 + 7 + 19 = 3^3$$
$$1 + 7 + 19 + 37 = 4^3$$

.

.

.

$$1 + 7 + 19 + \ldots + 217 = 9^3$$

Note that 2^3 is the sum of first 2 numbers of 1, 7, 19, 37, ... Similarly, $3^3, 4^3, \ldots, 9^3$ are the sums of first 3, 4, ..., 9 numbers respectively out of 1, 7, 19, ..., 217. These numbers may be obtained by putting $n = 1, 2, 3, \ldots$ in $1 + n \times (n - 1) \times 3$.

Thus, to find the cube root of a perfect cube, we go on subtracting

$1 (= 1 + 1 \times 0 \times 3)$, $7 (= 1 + 2 \times 1 \times 3)$, $19 (= 1 + 3 \times 2 \times 3)$, $37 (= 1 + 4 \times 3 \times 3)$, ... etc., from the given number till we get a zero. The number of times the subtraction is carried out, gives the cube root. For example,

$216 - 1 = 215,\ 215 - 7 = 208,\ 208 - 19 = 189,\ 189 - 37 = 152,\ 152 - 61 = 91,$
$91 - 91 = 0.$

Since we have subtracted six times to get 0, therefore $\sqrt[3]{216} = 6$.

This method may be used to find the cube roots of small numbers. This method may also be used to find the smallest number to be added or subtracted to make a non-perfect cube a perfect cube.

Example 6 : Examine if 400 is a perfect cube. If not, then find the smallest number that must be subtracted from 400 to obtain a perfect cube.

Solution : $400 - 1 = 399,\ 399 - 7 = 392,\ 392 - 19 = 373,\ 373 - 37 = 336,$
$336 - 61 = 275,\ 275 - 91 = 184,\ 184 - 127 = 57$

The next number to be subtracted is 169 which is greater than 57. Therefore, the process of successive subtraction does not give zero. Hence, 400 is not a perfect cube. If we subtract 57 from 400, the above process will give zero after 7 successive subtractions. Therefore, $400 - 57 = 7^3$.

Thus, 57 is the required number (and 343 is the resulting perfect cube). Similarly, if 112 is added to 400, the sum 512 is a perfect cube. (Why ?)

2.6 Cube Root Using Units Digit

We shall now describe a method that can be used to find cube roots of perfect cubes having at the most six digits. By looking at Table 2.2, we observe that the cube of a number ending in 0, 1, 4, 5, 6 and 9 ends in 0, 1, 4, 5, 6 and 9 respectively. However, the cube of a number ending in 2 ends in 8 and vice versa. Similarly, the cube of a number ending in 3 or 7 ends in 7 or 3 respectively. Thus, by looking at the units digit of a perfect cube number, we can determine the units digit of its cube root.

Now consider a number which is a perfect cube and has at the most six digits. The cube root of such a number has at the most two digits, because the least seven digit number is 1000000 ($= 100^3$) and its cube root 100 is a three digit number. We determine the two digits of the cube root as follows :

Step 1 : Look at the digit at the units place of the perfect cube and determine the digit at the units place in the cube root as discussed above.

Step 2 : Strike out from the right, last three (i.e., units, tens and hundreds) digits of the number. If nothing is left, we stop. The digit in Step 1 is the cube root.

Step 3 : Consider the number left from Step 2. Find the largest single digit number whose cube is less than or equal to this left over number. This is the tens digit of the cube root.

Example 7 : Find the cube roots of the following numbers :

(i) 512 (ii) 2197 (iii) 117649 (iv) 636056

Solution : (i) 512 : The units digit of 512 is 2. Therefore, the digit at the units place in the cube root is 8. Since no number is left after striking out the units, tens and hundreds digits of the number, the required cube root is 8.

(ii) 2197 : Here, units digit is 7. Therefore, units digit of the cube root is 3. After striking out the last three digits from the right, we are left with the number 2. Now 1 is the largest number whose cube is less than 2. Therefore, the tens digit is 1.

Thus, the required cube root is 13.

(iii) 117649 : Here, units digit is 9. Therefore, the units digit of the cube root is 9. Striking out the last three digits from the right, the number left is 117. Now $4^3 = 64 < 117$ and $5^3 = 125 > 117$.

Hence, the tens digit of the cube root is 4.

$\therefore \qquad \sqrt[3]{117649} = 49$

(iv) 636056 : Here, units digit of the cube root is 6. (Why ?)

Also, $8^3 < 636$ and $9^3 > 636$.

Hence, tens digit of the cube root is 8

$\therefore \qquad \sqrt[3]{636056} = 86$

2.7 Cube Root by Prime Factorisation

We have already observed that in the prime factorisation of a perfect cube, primes occur in triples. We, therefore, can find $\sqrt[3]{n}$ using the following algorithm :

1. Find the prime factorisation of n .
2. Group the factors in triples such that all three factors in each triple are the same.
3. If some prime factors are left ungrouped, the number n is not a perfect cube and the process stops.
4. If no factor is left ungrouped, choose one factor from each group and take their product. The product is the cube root of n.

Example 8 : Find the cube root of :

(i) 91125 (ii) 531441 (iii) 551368

Solution : (i) $91125 = \underline{5 \times 5 \times 5} \times \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3}$

5	91125
5	18225
5	3645
3	729
3	243
3	81
3	27
3	9
	3

$\therefore \quad \sqrt[3]{91125} = 5 \times 3 \times 3$

$= 45$

(ii) $531441 = \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3}$

3	531441
3	177147
3	59049
3	19683
3	6561
3	2187
3	729
3	243
3	81
3	27
3	9
	3

$\therefore \quad \sqrt[3]{531441} = 3 \times 3 \times 3 \times 3$

$= 81$

(iii) $551368 = \underline{2 \times 2 \times 2} \times \underline{41 \times 41 \times 41}$

2	551368
2	275684
2	137842
41	68921
41	1681
	41

$\therefore \quad \sqrt[3]{551368} = 2 \times 41$

$= 82$

2.8 Cube Roots of Negative Numbers

Consider the perfect cubes 27 ($= 3^3$) and 343 ($= 7^3$). We know that $27 \times 343 = 9261$ is also a perfect cube, because $a^3 \times b^3 = (a \times b)^3$ holds for integers a and b. Now we find the cube root of 9261 and observe that

$$\sqrt[3]{9261} = 21 = 3 \times 7 = \sqrt[3]{27} \times \sqrt[3]{343}$$

i.e.,
$$\sqrt[3]{27 \times 343} = \sqrt[3]{27} \times \sqrt[3]{343}$$

There is nothing special about 27 or 343. In fact, for any two perfect cubes x and y, we have

$$\sqrt[3]{x \times y} = \sqrt[3]{x} \times \sqrt[3]{y}$$

i.e., cube root of a product of two perfect cubes is the product of their cube roots.

Making use of this result, we find the cube roots of negative integers. For a positive integer m

$$-m = -1 \times m$$

$\therefore \quad \sqrt[3]{-m} = \sqrt[3]{-1} \times \sqrt[3]{m}$

However, $\sqrt[3]{-1} = -1$, because $(-1)^3 = -1$

$\therefore \quad \sqrt[3]{-m} = -\sqrt[3]{m}$

Example 9 : Find the cube roots of (i) – 125, (ii) – 343 and (iii) – 2197.

Solution : (i) $\sqrt[3]{-125} = -\sqrt[3]{125} = -5 \qquad (\because 5^3 = 125)$

(ii) $\sqrt[3]{-343} = -\sqrt[3]{343} = -7 \qquad (\because 7^3 = 343)$

(iii) $\sqrt[3]{-2197} = -\sqrt[3]{2197} = -13 \qquad (\because 13^3 = 2197)$

2.9 Cube Roots of Rational Numbers

Like the cube root of the product of two perfect cubes, we have the following result for the cube root of the quotient of two perfect cubes :

If x and y $(\neq 0)$ are perfect cubes, then $\sqrt[3]{\frac{x}{y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{y}}$, i.e., *the cube root of the quotient of two perfect cubes is the quotient of their cube roots.*

Note that $\sqrt[3]{x}$ and $\sqrt[3]{y}$ are integers and $\sqrt[3]{y} \neq 0$. Hence, $\frac{\sqrt[3]{x}}{\sqrt[3]{y}}$ is a rational number.

Thus, the cube root of a rational number whose numerator and denominator are perfect cubes, is also a rational number. The numerator of the cube root is the cube root of the numerator of the given number and the denominator of the cube root is the cube root of the denominator of the given number.

Example 10 : Find the cube roots of (i) $\frac{343}{125}$, (ii) $\frac{-27}{512}$ and (iii) $\frac{-2197}{1331}$.

Solution : (i) $\sqrt[3]{\frac{343}{125}} = \frac{\sqrt[3]{343}}{\sqrt[3]{125}} = \frac{\sqrt[3]{7 \times 7 \times 7}}{\sqrt[3]{5 \times 5 \times 5}} = \frac{7}{5}$

(ii) $\sqrt[3]{\frac{-27}{512}} = \frac{\sqrt[3]{-27}}{\sqrt[3]{512}} = \frac{-\sqrt[3]{3 \times 3 \times 3}}{\sqrt[3]{8 \times 8 \times 8}} = \frac{-3}{8} = -\frac{3}{8}$

(iii) $\sqrt[3]{\frac{-2197}{1331}} = \frac{-\sqrt[3]{2197}}{\sqrt[3]{1331}} = \frac{-\sqrt[3]{13 \times 13 \times 13}}{\sqrt[3]{11 \times 11 \times 11}} = \frac{-13}{11} = -\frac{13}{11}$

EXERCISE 2.2

1. Find the cube roots of the following numbers by successive subtraction of numbers 1, 7, 19, 37, 61, 91, 127, 169, 217, 271, 331, 397, . . . :

   (i) 64 (ii) 512 (iii) 1728

2. Using the method of Question 1, examine if the following numbers are perfect cubes :

   (i) 130 (ii) 345 (iii) 792 (iv) 1331

3. Find the smallest number that must be subtracted from those numbers in Question 2 which are not perfect cubes so as to make them perfect cubes. What are the corresponding cube roots?

4. Find the units digits of the cube roots of the following numbers :

   (i) 226981 (ii) 13824 (iii) 571787 (iv) 175616

5. Find the tens digit of the cube root of each of the numbers in Question 4.

6. Find the cube roots of the following numbers by finding their units and tens digits :

   (i) 389017 (ii) 91125 (iii) 110592 (iv) 46656

7. Find the cube roots of the following numbers using prime factorisation :

   (i) 250047 (ii) 438976 (iii) 592704 (iv) 614125

8. Find the cube roots of

   (i) – 226981 (ii) – 13824 (iii) – 571787 (iv) – 175616

9. Find the cube roots of the numbers 2460375, 20346417, 210644875, 57066625 using the fact that :

(i) $2460375 = 3375 \times 729$ (ii) $20346417 = 9261 \times 2197$

(iii) $210644875 = 42875 \times 4913$ (iv) $57066625 = 166375 \times 343$

[*Hint* : $a^3\ b^3 = (ab)^3$]

10. Find the cube roots of :

(i) $\frac{729}{2197}$ (ii) $\frac{3375}{4913}$ (iii) $\frac{9261}{42875}$ (iv) $\frac{343}{166375}$

11. Examine, whether or not each of the following numbers has a cube root. If not, find the smallest number by which the number must be multiplied so that the product has a cube root.

(i) 3087 (ii) 33275 (iii) 120393

12. Find the smallest number by which the numbers in Question 11 must be divided so that the quotient has a cube root.

Things to Remember

1. A number n is a perfect cube, if there is an integer m such that $n = m^3$.
2. If n is a perfect cube and $n = m^3$, then m is a cube root of n. A cube root of n is written as $\sqrt[3]{n}$.
3. The units digit of the cube root of a perfect cube can be determined with the help of the units digit of the perfect cube.
4. The cube root of a perfect cube can be obtained by prime factorisation of the number.
5. The cube root of a product of two perfect cubes is the product of the cube roots of the perfect cubes, i.e.,

   $\sqrt[3]{ab} = \sqrt[3]{a}\,.\sqrt[3]{b}$, where a and b are perfect cubes.
6. The cube root of a quotient of two perfect cubes is the quotient of their cube roots, i.e.,

   $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$, $b \neq 0$, where a and b are perfect cubes.
7. The cube root of a negative perfect cube is negative.

CHAPTER

3

RATIONAL EXPONENTS AND RADICALS

3.1 Introduction

Let us recall that if x is a rational number and m is a positive integer, then

$$x^m = x \times x \times \ldots \times x, \ m \text{ times}.$$

Similarly, if x is a non-zero rational number and k is a negative integral exponent, say $k = -m$, where m is a positive integer, then

$$\begin{aligned} x^k &= x^{-m} = x^{-1} \times x^{-1} \times \ldots \times x^{-1}, \ m \text{ times} \\ &= \frac{1}{x} \times \frac{1}{x} \times \ldots \times \frac{1}{x}, \ m \text{ times} \\ &= \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \end{aligned}$$

We also know that if x is a non-zero rational number and m and n are integers, then

$$\begin{aligned} x^m \times x^n &= x^{m+n} \\ x^m \div x^n &= x^{m-n} \\ (x^m)^n &= x^{m \times n} \end{aligned}$$

Also, $\quad x^m y^m = (xy)^m \qquad$ (x and y non-zero rational numbers)

In this Chapter, we give meaning to x^m, where x is a positive rational number and m is a rational exponent. We also obtain relations of the above type for rational exponents instead of integeral exponents.

3.2 Positive Rational Exponents

We know that $3^3 = 27$. We also express this relation as $27^{\frac{1}{3}} = 3$. Similarly, the relation

$2^5 = 32$ can also be expressed as $32^{\frac{1}{5}} = 2$. In general, if x and y are non-zero rational numbers and m is a positive integer such that $x^m = y$, then we may write $y^{\frac{1}{m}} = x$. We may also write $y^{\frac{1}{m}}$ as $\sqrt[m]{y}$ and call it the *m*th *root of* y. Thus,

$$\text{2nd root of } 4 = \sqrt[2]{4} = 2$$

$$\text{3rd root of } 27 = \sqrt[3]{27} = 3$$

$$\text{4th root of } 625 = \sqrt[4]{625} = 5, \text{ etc.}$$

Using this meaning, we may define x^m for a positive rational exponent m.

If x is a positive rational number, and $m = \frac{p}{q}$ is a positive rational exponent, then we define $x^{\frac{p}{q}}$ as the qth root of x^p,

i.e., $x^{\frac{p}{q}} = \left(x^p\right)^{\frac{1}{q}}$

For example,

$$8^{\frac{5}{3}} = \left(8^5\right)^{\frac{1}{3}} = (32768)^{\frac{1}{3}} = 32$$

Further, $\left(8^{\frac{1}{3}}\right)^5 = 2^5 = 32$

Thus, $\left(8^5\right)^{\frac{1}{3}} = \left(8^{\frac{1}{3}}\right)^5$.

There is nothing special about 8, 5 or 3 here. The result is true in general.

If x is a positive rational number, then for a positive rational exponent $\frac{p}{q}$,

$$\left(x^p\right)^{\frac{1}{q}} = \left(x^{\frac{1}{q}}\right)^p$$

We may thus define $x^{\frac{p}{q}}$ in any of the following equivalent forms :

(A): $x^{\frac{p}{q}} = \left(x^p\right)^{\frac{1}{q}} = \sqrt[q]{x^p}$, read as *the* qth *root of* x^p (the pth power of x)

(B): $x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = \left(\sqrt[q]{x}\right)^p$, read as *the* pth *power of the* qth *root of* x.

Example 1 : Find : (i) $8^{\frac{2}{3}}$ (ii) $\left(\frac{32}{243}\right)^{\frac{4}{5}}$

Solution : (i) $8^{\frac{2}{3}} = \left(8^2\right)^{\frac{1}{3}}$, using form (A)

$= (64)^{\frac{1}{3}}$

$= 4$, since $4^3 = 64$

or

$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$, using form (B)

$= (2)^2$, since $2^3 = 8$

$= 4$

(ii) $\left(\frac{32}{243}\right)^{\frac{4}{5}} = \left[\left(\frac{32}{243}\right)^4\right]^{\frac{1}{5}}$, using form (A)

$= \left[\left(\frac{2^5}{3^5}\right)^4\right]^{\frac{1}{5}}$, since $32 = 2^5,\ 243 = 3^5$

$= \left[\frac{\left(2^5\right)^4}{\left(3^5\right)^4}\right]^{\frac{1}{5}}$, using $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$= \left(\frac{2^{20}}{3^{20}}\right)^{\frac{1}{5}}$, using $(x^m)^n = x^{mn}$

$= \left[\left(\frac{2}{3}\right)^{20}\right]^{\frac{1}{5}}$, using $\frac{x^m}{y^m} = \left(\frac{x}{y}\right)^m$

$= \left(\frac{2}{3}\right)^4$, since $\left(\left(\frac{2}{3}\right)^4\right)^5 = \left(\frac{2}{3}\right)^{20}$

$= \frac{16}{81}$

or

$$\left(\frac{32}{243}\right)^{\frac{4}{5}} = \left[\left(\frac{32}{243}\right)^{\frac{1}{5}}\right]^4, \text{ using form (B)}$$

$$= \left(\left[\left(\frac{2}{3}\right)^5\right]^{\frac{1}{5}}\right)^4$$

$$= \left(\frac{2}{3}\right)^4, \text{ using } \left(x^m\right)^{\frac{1}{m}} = x,\ x > 0$$

$$= \frac{2^4}{3^4} = \frac{16}{81}$$

Remark : In the above example, we have used both the forms (A) and (B). Which form do you think is easier for the purpose of calculation?

Example 2 : Evaluate: (i) $\left(\frac{16}{81}\right)^{\frac{3}{4}} \times \left(\frac{16}{81}\right)^{\frac{5}{4}}$ (ii) $\left(\frac{16}{81}\right)^{\frac{3}{4}+\frac{5}{4}}$

Solution : (i) $\left(\frac{16}{81}\right)^{\frac{3}{4}} \times \left(\frac{16}{81}\right)^{\frac{5}{4}} = \left[\left(\frac{16}{81}\right)^{\frac{1}{4}}\right]^3 \times \left[\left(\frac{16}{81}\right)^{\frac{1}{4}}\right]^5$, using form (B)

$$= \left(\frac{2}{3}\right)^3 \times \left(\frac{2}{3}\right)^5, \text{ since } \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$= \left(\frac{2}{3}\right)^8, \text{ since for integers } m \text{ and } n,\ x^m \times x^n = x^{m+n}$$

$$= \frac{2^8}{3^8} = \frac{256}{6561}$$

(ii) $\left(\frac{16}{81}\right)^{\frac{3}{4}+\frac{5}{4}} = \left(\frac{16}{81}\right)^2$, since $\frac{3}{4}+\frac{5}{4} = 2$

$$= \frac{16^2}{81^2} = \frac{256}{6561}$$

Remark : From the above example, we observe that

$$\left(\frac{16}{81}\right)^{\frac{3}{4}} \times \left(\frac{16}{81}\right)^{\frac{5}{4}} = \left(\frac{16}{81}\right)^{\frac{3}{4}+\frac{5}{4}}$$

which follows the rule $x^m \times x^n = x^{m+n}$.

Example 3 : Evaluate :

(i) $\left(\frac{16}{81}\right)^{\frac{5}{4}} \div \left(\frac{16}{81}\right)^{\frac{3}{4}}$ (ii) $\left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}}$

Solution : (i) $\left(\frac{16}{81}\right)^{\frac{5}{4}} \div \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left[\left(\frac{16}{81}\right)^{\frac{1}{4}}\right]^5 \div \left[\left(\frac{16}{81}\right)^{\frac{1}{4}}\right]^3$

$= \left(\frac{2}{3}\right)^5 \div \left(\frac{2}{3}\right)^3$

$= \left(\frac{2}{3}\right)^2$, since for integers m and n, $x^m \div x^n = x^{m-n}$

$= \frac{2^2}{3^2} = \frac{4}{9}$

(ii) $\left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{1}{2}}$, since $\frac{5}{4} - \frac{3}{4} = \frac{1}{2}$

$= \frac{4}{9}$, since $\left(\frac{4}{9}\right)^2 = \frac{16}{81}$

Remark : From the above example, we observe that

$$\left(\frac{16}{81}\right)^{\frac{5}{4}} \div \left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{16}{81}\right)^{\frac{5}{4}-\frac{3}{4}}$$

which follows the rule $x^m \div x^n = x^{m-n}$.

3.3 Negative Rational Exponents

Recall from Class VII that if m is a positive integer and x is a non-zero rational number, then $x^{-m} = \frac{1}{x^m} = \left(\frac{1}{x}\right)^m$.

i.e., x^{-m} is the *reciprocal of* x^m or *the mth power of the reciprocal of x.* We adopt the same rule for rational exponents also. If $\frac{p}{q}$ is a positive rational number and $x > 0$ is a rational number, then

$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}} = \left(\frac{1}{x}\right)^{\frac{p}{q}},$$

i.e., $x^{-\frac{p}{q}}$ *is the reciprocal of* $x^{\frac{p}{q}}$ *or the number obtained by raising the reciprocal of x to the exponent* $\frac{p}{q}$. Thus, if $x = \frac{r}{s}$ $(r, s > 0)$, then $\left(\frac{r}{s}\right)^{-\frac{p}{q}} = \left(\frac{s}{r}\right)^{\frac{p}{q}}$, since the reciprocal of $\frac{r}{s}$ is $\frac{s}{r}$.

Example 4 : Find : (i) $8^{-\frac{2}{3}}$ (ii) $\left(\frac{32}{243}\right)^{-\frac{4}{5}}$

Solution : (i) $8^{-\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$

$$= \left[\left(\frac{1}{8}\right)^{\frac{1}{3}}\right]^2 = \left(\frac{1}{2}\right)^2, \text{ since } \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$= \frac{1}{4}$$

(ii) $\left(\frac{32}{243}\right)^{-\frac{4}{5}} = \left(\frac{243}{32}\right)^{\frac{4}{5}} = \left[\left(\frac{243}{32}\right)^{\frac{1}{5}}\right]^4$

$$= \left[\left(\frac{3^5}{2^5}\right)^{\frac{1}{5}}\right]^4 = \left[\left(\left(\frac{3}{2}\right)^5\right)^{\frac{1}{5}}\right]^4$$

$$= \left(\frac{3}{2}\right)^4 = \frac{81}{16}$$

Example 5 : Evaluate :

(i) $\left(\frac{27}{125}\right)^{\frac{-2}{3}} \times \left(\frac{27}{125}\right)^{\frac{-4}{3}}$ (ii) $\left(\frac{27}{125}\right)^{\frac{-2}{3}+\left(\frac{-4}{3}\right)}$

Solution :(i) $\left(\frac{27}{125}\right)^{\frac{-2}{3}} \times \left(\frac{27}{125}\right)^{\frac{-4}{3}} = \left(\frac{125}{27}\right)^{\frac{2}{3}} \times \left(\frac{125}{27}\right)^{\frac{4}{3}}$

$$= \left[\left(\frac{5^3}{3^3}\right)^{\frac{1}{3}}\right]^2 \times \left[\left(\frac{5^3}{3^3}\right)^{\frac{1}{3}}\right]^4 = \left[\left(\left(\frac{5}{3}\right)^3\right)^{\frac{1}{3}}\right]^2 \times \left[\left(\left(\frac{5}{3}\right)^3\right)^{\frac{1}{3}}\right]^4$$

$$= \left(\frac{5}{3}\right)^2 \times \left(\frac{5}{3}\right)^4 = \left(\frac{5}{3}\right)^6 = \frac{15625}{729}$$

(ii) $\left(\frac{27}{125}\right)^{\frac{-2}{3}+\left(\frac{-4}{3}\right)} = \left(\frac{27}{125}\right)^{\frac{-6}{3}}$

$$= \left(\frac{27}{125}\right)^{-2}$$

$$= \left(\frac{125}{27}\right)^{2} = \frac{15625}{729}$$

Remark : From the above example, we observe that

$$\left(\frac{27}{125}\right)^{\frac{-2}{3}} \times \left(\frac{27}{125}\right)^{\frac{-4}{3}} = \left(\frac{27}{125}\right)^{\frac{-2}{3}+\left(\frac{-4}{3}\right)}$$

which follows the rule $x^m \times x^n = x^{m+n}$.

Example 6 : Evaluate the following expressions and examine if they are equal:

(i) $\left(\frac{27}{125}\right)^{\frac{-2}{3}} \div \left(\frac{27}{125}\right)^{\frac{-4}{3}}$ (ii) $\left(\frac{27}{125}\right)^{\frac{-2}{3}-\left(\frac{-4}{3}\right)}$

Solution : (i) $\left(\frac{27}{125}\right)^{\frac{-2}{3}} \div \left(\frac{27}{125}\right)^{\frac{-4}{3}} = \left(\frac{125}{27}\right)^{\frac{2}{3}} \div \left(\frac{125}{27}\right)^{\frac{4}{3}}$

$$= \left[\left(\frac{5^3}{3^3}\right)^{\frac{1}{3}}\right]^2 \div \left[\left(\frac{5^3}{3^3}\right)^{\frac{1}{3}}\right]^4$$

$$= \left[\left(\left(\frac{5}{3}\right)^3\right)^{\frac{1}{3}}\right]^2 \div \left[\left(\left(\frac{5}{3}\right)^3\right)^{\frac{1}{3}}\right]^4$$

$$= \left(\frac{5}{3}\right)^2 \div \left(\frac{5}{3}\right)^4 = \left(\frac{5}{3}\right)^{2-4}$$

$$= \left(\frac{5}{3}\right)^{-2} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$$

(ii) $$\left(\frac{27}{125}\right)^{-\frac{2}{3}-\left(-\frac{4}{3}\right)} = \left(\frac{27}{125}\right)^{-\frac{2}{3}+\frac{4}{3}}$$

$$= \left(\frac{27}{125}\right)^{\frac{2}{3}} = \left[\left(\frac{3^3}{5^3}\right)^{\frac{1}{3}}\right]^2$$

$$= \left[\left(\left(\frac{3}{5}\right)^3\right)^{\frac{1}{3}}\right]^2 = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$$

Yes, the two values are equal.

Remark : The above example shows that

$$\left(\frac{27}{125}\right)^{-\frac{2}{3}} \div \left(\frac{27}{125}\right)^{-\frac{4}{3}} = \left(\frac{27}{125}\right)^{-\frac{2}{3}-\left(-\frac{4}{3}\right)}$$

which follows the rule $x^m \div x^n = x^{m-n}$.

3.4 Laws of Exponents

We know that if x is a non-zero rational number and m and n are integral exponents, then

$$x^m \times x^n = x^{m+n} \qquad (1)$$

$$x^m \div x^n = x^{m-n} \qquad (2)$$

$$\left(x^m\right)^n = x^{m \times n} \qquad (3)$$

Further, if x and y are non-zero rational numbers, then

$$x^m \times y^m = (x \times y)^m \qquad (4)$$

The above relations are also true when m and n are rational exponents and, x and y are positive rational numbers.

Examples 2 and 5 illustrate the following law :

Law (1) : If $x > 0$ is a rational number and m and n are rational exponents, then

$$x^m \times x^n = x^{m+n}$$

Similarly, Examples 3 and 6 illustrate the following law :

Law (2) : For a rational number $x > 0$ and rational exponents m and n,

$$x^m \div x^n = x^{m-n}$$

Both the above laws are valid even when one of the exponents is positive and the other negative.

Let us verify relation (3) for rational exponents. For this let us consider the following examples :

Example 7 : Evaluate (i) $\left[\left(\frac{25}{9}\right)^{\frac{5}{2}}\right]^{\frac{3}{5}}$ and (ii) $\left(\frac{25}{9}\right)^{\frac{5}{2}\times\frac{3}{5}}$ and show that their values are equal.

Solution : (i) $$\left[\left(\frac{25}{9}\right)^{\frac{5}{2}}\right]^{\frac{3}{5}} = \left[\left\{\left(\frac{5^2}{3^2}\right)^{\frac{1}{2}}\right\}^5\right]^{\frac{3}{5}}$$

$$= \left[\left\{\left(\left(\frac{5}{3}\right)^2\right)^{\frac{1}{2}}\right\}^5\right]^{\frac{3}{5}}$$

$$= \left[\left(\frac{5}{3}\right)^5\right]^{\frac{3}{5}} = \left[\left\{\left(\frac{5}{3}\right)^5\right\}^{\frac{1}{5}}\right]^3$$

$$= \left(\frac{5}{3}\right)^3 = \frac{5^3}{3^3} = \frac{125}{27}$$

(ii) $$\left(\frac{25}{9}\right)^{\frac{5}{2}\times\frac{3}{5}} = \left(\frac{25}{9}\right)^{\frac{3}{2}}$$

$$= \left[\left\{\left(\frac{5}{3}\right)^2\right\}^{\frac{1}{2}}\right]^3$$

$$= \left(\frac{5}{3}\right)^3 = \frac{5^3}{3^3} = \frac{125}{27}$$

Yes, the two values are equal.

Example 8 : Verify that $\left[(729)^{\frac{-5}{3}}\right]^{-\frac{1}{2}} = (729)^{-\frac{5}{3}\times\left(-\frac{1}{2}\right)}$.

Solution : $\left[(729)^{\frac{-5}{3}}\right]^{-\frac{1}{2}} = \left[\left(\frac{1}{729}\right)^{\frac{5}{3}}\right]^{-\frac{1}{2}} = \left[\left\{\left(\frac{1}{9^3}\right)^{\frac{1}{3}}\right\}^5\right]^{-\frac{1}{2}}$

$$= \left[\left(\frac{1}{9}\right)^5\right]^{-\frac{1}{2}} = \left(\frac{1}{9^5}\right)^{-\frac{1}{2}} = \left(\frac{9^5}{1}\right)^{\frac{1}{2}}$$

$$= \left[(3^2)^5\right]^{\frac{1}{2}} = (3^{10})^{\frac{1}{2}}$$

$$= 3^5 = 243$$

Also, $\quad (729)^{\frac{-5}{3}\times\left(-\frac{1}{2}\right)} = (729)^{\frac{5}{6}} = (3^6)^{\frac{5}{6}}$

$$= \left[(3^6)^{\frac{1}{6}}\right]^5 = 3^5 = 243$$

Thus, $\quad \left[(729)^{\frac{-5}{3}}\right]^{-\frac{1}{2}} = (729)^{\frac{-5}{3}\times\left(-\frac{1}{2}\right)}$.

The above are examples of the following law :

Law (3) : If x is a rational number, $x > 0$ and m and n are rational exponents, then

$$(x^m)^n = x^{m\times n}$$

We now look at an example of relation (4) stated earlier for rational exponents :

$$\left(\frac{8}{125}\right)^{\frac{2}{3}} \times \left(\frac{64}{27}\right)^{\frac{2}{3}} = \left[\left(\frac{2^3}{5^3}\right)^{\frac{1}{3}}\right]^2 \times \left[\left(\frac{4^3}{3^3}\right)^{\frac{1}{3}}\right]^2$$

$$= \left[\left(\left(\frac{2}{5}\right)^3\right)^{\frac{1}{3}}\right]^2 \times \left[\left(\left(\frac{4}{3}\right)^3\right)^{\frac{1}{3}}\right]^2$$

$$= \left(\frac{2}{5}\right)^2 \times \left(\frac{4}{3}\right)^2$$

$$= \left(\frac{2}{5} \times \frac{4}{3}\right)^2 = \left(\frac{8}{15}\right)^2$$

$$= \frac{64}{225}$$

and $$\left(\frac{8}{125} \times \frac{64}{27}\right)^{\frac{2}{3}} = \left[\left(\frac{8 \times 64}{125 \times 27}\right)^{\frac{1}{3}}\right]^2 = \left[\left(\frac{8^3}{5^3 \times 3^3}\right)^{\frac{1}{3}}\right]^2$$

$$= \left[\left(\left(\frac{8}{5 \times 3}\right)^3\right)^{\frac{1}{3}}\right]^2 = \left(\frac{8}{15}\right)^2$$

$$= \frac{64}{225}$$

Similarly,

$$(27)^{-\frac{1}{3}} \times \left(\frac{64}{729}\right)^{-\frac{1}{3}} = \left(\frac{1}{27}\right)^{\frac{1}{3}} \times \left(\frac{729}{64}\right)^{\frac{1}{3}}$$

$$= \left(\frac{1}{3^3}\right)^{\frac{1}{3}} \times \left(\left(\frac{9}{4}\right)^3\right)^{\frac{1}{3}}$$

$$= \frac{1}{3} \times \frac{9}{4} = \frac{3}{4}$$

and $$\left(27 \times \frac{64}{729}\right)^{-\frac{1}{3}} = \left(\frac{64}{27}\right)^{-\frac{1}{3}} = \left(\frac{27}{64}\right)^{\frac{1}{3}} = \left[\left(\frac{3}{4}\right)^3\right]^{\frac{1}{3}} = \frac{3}{4}.$$

Thus, we have *verifed* the following law :

Law (4) : If x and y are rational numbers, $x, y > 0$ and m is a rational exponent, then

$$x^m \times y^m = (x \times y)^m$$

Henceforth, we shall assume that the relations (1) to (4) mentioned in Section 3.4 are true for all positive rational bases and for all rational exponents.

Example 9 : Evaluate : (i) $(0.125)^{\frac{2}{3}}$ (ii) $(0.000729)^{\frac{-3}{4}} \times (0.09)^{\frac{-3}{4}}$

Solution : (i) $(0.125)^{\frac{2}{3}} = \left(\frac{125}{1000}\right)^{\frac{2}{3}}$

$$= \left(\frac{5^3}{10^3}\right)^{\frac{2}{3}} = \left[\left(\frac{5}{10}\right)^3\right]^{\frac{2}{3}}$$

$$= \left(\frac{5}{10}\right)^{3\times\frac{2}{3}}, \text{ using Law (3)}$$

$$= \left(\frac{5}{10}\right)^2 = \frac{5^2}{10^2}$$

$$= \frac{25}{100} = 0.25$$

(ii) $(0.000729)^{\frac{-3}{4}} \times (0.09)^{\frac{-3}{4}} = \left(\frac{729}{1000000}\right)^{\frac{-3}{4}} \times \left(\frac{9}{100}\right)^{\frac{-3}{4}}$

$$= \left(\frac{1000000}{729}\right)^{\frac{3}{4}} \times \left(\frac{100}{9}\right)^{\frac{3}{4}}$$

$$= \left(\frac{10^6}{9^3}\right)^{\frac{3}{4}} \times \left(\frac{10^2}{9}\right)^{\frac{3}{4}}$$

$$= \left(\frac{10^6 \times 10^2}{9^3 \times 9}\right)^{\frac{3}{4}}, \text{ using Law (4)}$$

$$= \left(\frac{10^8}{9^4}\right)^{\frac{3}{4}} = \frac{10^{8\times\frac{3}{4}}}{9^{4\times\frac{3}{4}}}$$

$$= \frac{10^6}{9^3} = \frac{1000000}{729}$$

Example 10 : Evaluate : (i) $\left(13^2 - 5^2\right)^{\frac{3}{2}}$ (ii) $\left(1^3 + 2^3 + 3^3 + 4^3\right)^{-\frac{3}{2}}$

Solution : (i) $\left(13^2 - 5^2\right)^{\frac{3}{2}} = \left[(13+5)\times(13-5)\right]^{\frac{3}{2}}$, $[x^2 - a^2 = (x+a)(x-a)]$

$$= (18\times 8)^{\frac{3}{2}} = (3\times 3\times 2\times 2\times 2\times 2)^{\frac{3}{2}}$$

$$= \left(3^2 \times 2^4\right)^{\frac{3}{2}} = \left(3^2\right)^{\frac{3}{2}} \times \left(2^4\right)^{\frac{3}{2}}, \qquad \text{[Law (4)]}$$

$$= 3^{2\times\frac{3}{2}} \times 2^{4\times\frac{3}{2}}, \qquad \text{[Law (3)]}$$

$$= 3^3 \times 2^6 = 1728$$

(ii) $\left(1^3 + 2^3 + 3^3 + 4^3\right)^{-\frac{3}{2}} = (1+8+27+64)^{-\frac{3}{2}}$

$$= (100)^{-\frac{3}{2}} = \left(10^2\right)^{-\frac{3}{2}}$$

$$= 10^{2\times\left(\frac{-3}{2}\right)}, \qquad \text{[Law (3)]}$$

$$= 10^{-3} = \frac{1}{1000}$$

3.5 Radicals and Radicands

We know that if $y > 0$ and $y^{\frac{1}{q}} = x$, then we also write $x = \sqrt[q]{y}$, i.e., $y^{\frac{1}{q}}$ and $\sqrt[q]{y}$ are two notations for the same expression. The form $y^{\frac{1}{q}}$ is called the *exponential form* and if q is positive, the form $\sqrt[q]{y}$ is called the *radical form*. The sign, $\sqrt{\ }$ is called the *radical sign* and $\sqrt[q]{y}$ is called a *radical*. The number q is called *the index of the radical* and y is called the *radicand*. Note that the index of a radical is always a positive integer. If q is negative, as in $32^{-\frac{1}{5}}$, then we do not write it as $\sqrt[-5]{32}$. We write $32^{-\frac{1}{5}}$ as $\left(\frac{1}{32}\right)^{\frac{1}{5}}$, i.e., as $\sqrt[5]{\frac{1}{32}}$. Here 5 is

the index of the radical and $\frac{1}{32}$ is the radicand.

Example 11 : Express : (i) $\sqrt[5]{1234}$ in the exponential form.

(ii) $\left(\frac{567}{890}\right)^{-\frac{1}{8}}$ as a radical.

Solution : (i) The required exponential form is $(1234)^{\frac{1}{5}}$.

(ii) $\left(\frac{567}{890}\right)^{\frac{-1}{8}} = \left(\frac{890}{567}\right)^{\frac{1}{8}}$

$\therefore$ The required radical form is $\sqrt[8]{\frac{890}{567}}$.

EXERCISE 3.1

1. Find the value of each of the following :

(i) $(16)^{\frac{1}{2}}$ (ii) $(243)^{\frac{1}{5}}$ (iii) $(15625)^{\frac{1}{6}}$

2. Evaluate : (i) $(32768)^{\frac{1}{15}}$ (ii) $(279936)^{\frac{1}{7}}$

3. Find the value of : (i) $\left(\frac{625}{81}\right)^{\frac{1}{4}}$ (ii) $\left(\frac{343}{1331}\right)^{\frac{1}{3}}$

4. Evaluate : (i) $\left(\frac{390625}{6561}\right)^{\frac{1}{8}}$ (ii) $\left(\frac{117649}{1771561}\right)^{\frac{1}{6}}$

5. Express the following in the exponential form:

(i) $\sqrt{5}$ (ii) $\sqrt[3]{7}$ (iii) $\sqrt[9]{1100}$ (iv) $\sqrt[4]{\frac{3}{4}}$ (v) $\sqrt[8]{\frac{61}{1123}}$

6. Express the following as radicals. In each case, find the radicand and the index of the radical.

(i) $16^{\frac{1}{2}}$ (ii) $125^{\frac{1}{3}}$ (iii) $\left(\frac{6}{17}\right)^{\frac{1}{9}}$ (iv) $\left(\frac{23}{11}\right)^{-\frac{1}{11}}$ (v) $\left(\frac{328}{61}\right)^{-\frac{1}{17}}$

7. Evaluate each of the following as the qth root of x^p, i.e., by the formula $x^{\frac{p}{q}} = \left(x^p\right)^{\frac{1}{q}}$:

(i) $8^{\frac{5}{3}}$ (ii) $\left(\frac{81}{16}\right)^{\frac{3}{4}}$ (iii) $\left(\frac{25}{49}\right)^{\frac{7}{2}}$ (iv) $\left(\frac{256}{6561}\right)^{\frac{3}{8}}$

8. Evaluate the expressions in Question 7 as the pth power of the qth root of x, i.e, by the formula $x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p$.

9. Evaluate: (i) $343^{-\frac{1}{3}}$ (ii) $\left(\frac{625}{81}\right)^{-\frac{1}{4}}$

10. Evaluate: (i) $\left(\frac{25}{81}\right)^{-\frac{3}{2}}$ (ii) $\left(\frac{256}{6561}\right)^{-\frac{5}{8}}$

11. Simplify each of the following :

(i) $23^{\frac{1}{2}} \times 23^{\frac{3}{2}}$ (ii) $11^{-\frac{4}{3}} \times 11^{-\frac{5}{3}}$

(iii) $3 \times 9^{\frac{3}{2}} \times 9^{-\frac{1}{2}}$ (iv) $27^{\frac{2}{3}} \times 27^{\frac{1}{3}} \times 27^{-\frac{4}{3}}$

12. Simplify each of the following :

(i) $15^{\frac{3}{2}} \div \left(\frac{1}{15}\right)^{\frac{1}{2}}$ (ii) $\left(\frac{2}{13}\right)^{\frac{4}{3}} \div \left(\frac{2}{13}\right)^{\frac{5}{3}}$

(iii) $3 \times 9^{\frac{1}{2}} \div 9^{\frac{3}{2}}$ (iv) $27^{\frac{2}{3}} \div 27^{\frac{1}{3}} \times 27^{-\frac{4}{3}}$

13. Evaluate each of the following :

(i) $(0.04)^{\frac{3}{2}}$ (ii) $(0.008)^{\frac{2}{3}}$ (iii) $(6.25)^{\frac{3}{2}}$ (iv) $(0.000064)^{\frac{5}{6}}$

14. Evaluate each of the following :

(i) $(3^2 + 4^2)^{-\frac{1}{2}}$ (ii) $(5^2 + 12^2)^{\frac{3}{2}}$

(iii) $(17^2 - 8^2)^{\frac{1}{2}}$ (iv) $(1^3 + 2^3 + 3^3)^{-\frac{5}{2}}$

15. Write true (T) or false (F) for the following statements :

(i) $\sqrt{x}$ is a rational number, if x is a perfect square.

(ii) If x is a negative rational number, then $\sqrt[3]{x^3} = x$ is not true.

(iii) For every integer x, $x^{\frac{3}{2}}$ is a rational number.

(iv) The exponential form of $\sqrt[p]{x^q}$ is $x^{\frac{p}{q}}$.

(v) The radical form of $\left(x^{\frac{1}{p}}\right)^{\frac{1}{q}}$ is $\sqrt[pq]{x}$.

(vi) $(x^{-3})^{-4} = x^{12}$ for every rational number $x > 0$.

(vii) Index of a radical is never negative.

Things to Remember

1. If m is a positive integer and x and y are rational numbers such that $x^m = y$, then $y^{\frac{1}{m}} = x$.

2. $y^{\frac{1}{m}}$ is called the mth root of y and is written as $\sqrt[m]{y}$.

3. If $x = \frac{r}{s}$ is a non-zero rational number, then $\left(\frac{r}{s}\right)^{-m} = \left(\frac{s}{r}\right)^{m}$.

4. For a rational exponent $\frac{p}{q}$,

$$x^{\frac{p}{q}} = \sqrt[q]{x^p}, \text{ the } q\text{th root of } x^p$$

or

$$x^{\frac{p}{q}} = (\sqrt[q]{x})^p, \text{ the } p\text{th power of the } q\text{th root of } x.$$

5. If $x = \frac{r}{s}$ is a positive rational number, then $x^{-\frac{p}{q}} = \left(\frac{r}{s}\right)^{-\frac{p}{q}} = \left(\frac{s}{r}\right)^{\frac{p}{q}}$.

6. If x is a positive rational number and m and n are any rational exponents, then
   (i) $x^m \times x^n = x^{m+n}$
   (ii) $x^m \div x^n = x^{m-n}$
   (iii) $(x^m)^n = x^{m \times n}$

7. If x and y are positive rational numbers and m is any rational exponent, then
$$x^m \times y^m = (x \times y)^m$$

8. If $x = \sqrt[q]{y} = y^{\frac{1}{q}}$, then $y^{\frac{1}{q}}$ is the exponential form and $\sqrt[q]{y}$ is the radical form of x. Here, q is the index of the radical and y is the radicand.

As History Tells Us

It is well known that all old civilizations had some kind of numeration system and had contributed to the development of different number systems. Ancient tablets found in Babylonia contain the squares of numbers from 1 to 60 and cubes of numbers from 1 to 32. Their period is believed to be of 2100 B.C. The word 'Square' here has its origin in the fact that the product of two equal numbers may be regarded as the area of a square with side equal to the numerical value of the number being multiplied by itself. Similarly, the use of the word cube in a^3 has similar interpretation. The current *notation* for integral exponents is due to *Descartes* (1637). He said : We *write aa or a^2 for multiplying a by itself and a^3 for multiplying the product once more by a and so on infinitely*. However, the theory of general exponents was understood much earlier.

The word square root has an intuitive meaning. Since $a \times a = a^2$, it is natural to label a as the *root of the square a^2* or *square root of a^2*. Similarly, for the word cube root. The method of finding square roots as given in *The Elements* (a work that compiled all mathematics known to Greeks till then) is quite similar to the one used today. *Theon of Alexandria* (390) gave a method to find square root using sexagesimal (numbers with base 60) system, which is more or less the same as our method. *Bhaskara* (1150) gave a method somewhat like Theon. In the earliest printed books on Arithmetic, square root is obtained by arranging numbers somewhat as in the galley method of division. It was during the 16^{th} and 17^{th} centuries that the currently used method became popular due to the efforts of *Cataneo* (1546) and *Cataldi* (1613).

Finding cube roots is a much more difficult problem than finding square roots. The methods for finding the square root were based on the geometric considerations of the simple formula $(a + b)^2 = a^2 + 2ab + b^2$. Naturally, finding cube roots depended on the relatively complex relation $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. The problem of finding cube roots was attempted by many ancient mathematicians. The names of *Brahmagupta* (628), *al-Karkhi* (1020), *Bhaskara* (1150) and *Fibonacci* (1202) may be mentioned in this connection. During the 16^{th} and the 17^{th} centuries, people explained the process of extracting cube roots by means of cubical blocks taking x cubes consisting of x^2 cubes each. (Notice $x^3 = x \times x^2$, etc.).

As early as 1360, the French mathematician *Nicole Oresme* and other mathematicians were considering exponents of the form $\frac{1}{2}$, but it was during the 17^{th} century only that *Wallis* (1655) explained the theory of exponents satisfactorily. He considered fractional and negative exponents. He suggested that x^0 should be taken as 1 and $x^{p/q}$ should be treated as $\sqrt[q]{x^p}$. *Newton* (1669) further modified Wallis' work. After Newton, the current notation of exponents got established.

CHAPTER

4

PROFIT, LOSS AND DISCOUNT

4.1 Introduction

In Class VI, we started the study of *Profit and Loss* as an application of percentage. We extended this study by including problems involving *overhead charges* in Class VII. In this Chapter, we shall learn more about profit and loss. We shall also introduce the concept of *discount* at an elementary level and will solve simple problems related to it.

4.2 Profit and Loss

You are already familiar with profit and loss from your earlier classes. Recall that if the selling price (S.P.) of an article is greater than its cost price (C.P.), we say that there is a *profit*. On the other hand, if S.P. of an article is less than its C.P., we say that there is a *loss*. We may list the various relations regarding profit and loss as follows :

1. In case of profit or gain (i.e., if S.P. > C.P.),

   (i) $\text{Profit} = \text{S.P.} - \text{C.P.}$

   (ii) $\text{S.P.} = \text{Profit} + \text{C.P.}$

   (iii) $\text{C.P.} = \text{S.P.} - \text{Profit}$

   (iv) $\text{Profit }\% = \dfrac{\text{Profit}}{\text{C.P.}} \times 100$

   (v) $\text{Profit} = \dfrac{\text{C.P.} \times \text{Profit }\%}{100}$ [From (iv) above]

   (vi) $\text{S.P.} = \text{C.P.} \times \left(\dfrac{100 + \text{Profit }\%}{100}\right)$ [From (ii) and (v) above]

   (vii) $\text{C.P.} = \dfrac{100 \times \text{S.P.}}{100 + \text{Profit }\%}$ [From (vi) above]

2. In case of loss (i.e., if S.P. < C.P.),

(i) Loss = C.P. – S.P.

(ii) S.P. = C.P. – Loss

(iii) C.P. = Loss + S.P.

(iv) $\text{Loss } \% = \frac{\text{Loss}}{\text{C. P.}} \times 100$

(v) $\text{Loss} = \frac{\text{C. P.} \times \text{Loss } \%}{100}$ [From (iv) above]

(vi) $\text{S.P.} = \text{C.P.} \times \left(\frac{100 - \text{Loss } \%}{100}\right)$ [From (ii) and (v) above]

(vii) $\text{C.P.} = \frac{100 \times \text{S. P.}}{100 - \text{Loss } \%}$ [From (vi) above]

It must always be kept in mind that *overhead charges* such as transportation charges, expenditure on repairs, etc. (if any) are always included in the C.P.. We now consider some more examples related to profit and loss.

Example 1 : Anwar purchased 120 oranges at the rate of Rs 2 per orange. He sold 60 % of the oranges at the rate of Rs 2.50 per orange and the remaining oranges at the rate of Rs 2 per orange. Find his profit per cent.

Solution : C.P. of 120 oranges = Rs 2 × 120 = Rs 240

60 % of 120 oranges = $\frac{60}{100} \times 120$ oranges = 72 oranges

Now S.P. of 72 oranges = Rs 2.50 × 72 = Rs 180

and S.P. of the remaining 120 – 72, i.e., 48 oranges = Rs 2 × 48 = Rs 96

∴ S.P. of all the 120 oranges = Rs 180 + Rs 96 = Rs 276

Therefore, profit = S.P. – C.P.

= Rs (276 – 240) = Rs 36

Hence, profit per cent = $\frac{36}{240} \times 100 = 15$

Thus, Anwar's profit is 15 %.

Example 2 : Maninder bought two horses at Rs 20000 each. He sold one horse at 15 % gain. But he had to sell the second horse at a loss. If he had suffered a loss of Rs 1800 on the whole transaction, find the selling price of the second horse.

Solution : Total C.P. of the two horses = 2 × Rs 20000 = Rs 40000

Loss = Rs 1800

∴ Total S.P. = Rs 40000 – Rs 1800 = Rs 38200 (1)

Now S.P. of the first horse at 15 % profit = C.P. $\left(\frac{100 + \text{Profit \%}}{100}\right)$

$$= \text{Rs } 20000 \frac{(100 + 15)}{100} = \text{Rs } 23000 \quad (2)$$

∴ S.P. of the second horse = Rs 38200 – Rs 23000 [From (1) and (2)]

= Rs 15200

Thus, the selling price of the second horse is Rs 15200.

Example 3 : By selling 144 hens, Malleshwari lost the S.P. of 6 hens. Find her loss per cent.

Solution : Let S.P. of 1 hen = Re 1

∴ S.P. of 144 hens = Rs 144 × 1 = Rs 144

and Loss = S.P. of 6 hens

= Rs 6 × 1 = Rs 6

∴ C.P. of 144 hens = S.P. + Loss

= Rs 144 + Rs 6 = Rs 150

Therefore, loss % = $\frac{\text{Loss}}{\text{C. P.}} \times 100 = \frac{6}{150} \times 100 = 4$

Thus, Malleshwari's loss is 4 %.

Example 4 : A farmer sold two bullocks for Rs 18000 each. On one bullock he gained 20 % and on the other he lost 20 %. Find his total loss or gain.

Solution : S.P. of the first bullock = Rs 18000

Gain = 20%

Therefore, C.P. = $\frac{100 \times \text{S. P.}}{100 + \text{Profit \%}}$

$$= \text{Rs} \frac{100 \times 18000}{100 + 20} = \text{Rs } 15000 \quad (1)$$

S.P. of the second bullock = Rs 18000

Loss = 20%

Therefore, $\text{C.P.} = \dfrac{100 \times \text{S. P.}}{100 - \text{Loss }\%}$

$= \text{Rs } \dfrac{100 \times 18000}{100 - 20} = \text{Rs } 22500$ (2)

Now, total C.P. = Rs 15000 + Rs 22500 [From (1) and (2)]

= Rs 37500

and total S.P. = 2 × Rs 18000 = Rs 36000

Hence, loss = C.P. – S.P.

= Rs 37500 – Rs 36000

= Rs 1500

EXERCISE 4.1

1. A shopkeeper purchased 100 blankets at Rs 2000 each. He found that 10 blankets were defective and he sold these at Rs 1200 each. At what rate should he sell the remaining blankets so as to gain 14 % on the whole ?

2. Sarita and Salma purchased one buffalo each for the same price. Sarita sold it for Rs 14880 and lost 7 %. At what price should Salma sell her buffalo so as to earn a profit of 5 %?

3. Gorang bought 60 kg of apples at Rs 48 per kg. He sold 70 % of the apples at Rs 60 per kg and the remaining apples at Rs 35 per kg. Find Gorang's gain or loss per cent on the whole transaction.

4. A goldsmith bought 100 g of gold for Rs 54000 from a wholesaler. He then sold it at a gain of 10 %. Find the
   (i) S.P. of 10 g of gold for the goldsmith.
   (ii) C.P. of 10 g of gold for the wholesaler, if his profit is 8 %.

5. William sells a quintal of wheat for Rs 924 and earns a profit of 12 %. By selling a quintal of rice for the same amount he loses 12 %. Find
   (i) C.P. of wheat.
   (ii) C.P. of rice.
   (iii) gain or loss per cent on the whole transaction.

6. A tricycle is sold at a gain of 16 %. Had it been sold for Rs 100 more, the gain would have been 20 %. Find the C.P. of the tricycle.
   [*Hint* : Let C.P. be Rs 100. Difference between the two selling prices
   = Rs (120 – 116) = Rs 4, etc.]

7. Ahmed purchased a radio set for Rs 2700 and spent Rs 300 on its repairs. He then sold it to Karim at a gain of 25 %. Karim sold it to Subramaniam at a loss of 10 %. Find the C.P. for

   (i) Karim (ii) Subramaniam

8. Utkarsh bought 20 dining tables for Rs 120000 and sold these at a profit equal to the selling price of 4 dining tables. Find the S.P. of one dining table.

9. Yogita sold a plot of land at 20 % gain to Dimpi. Dimpi sold it to Anjali at 10 % profit. If Anjali had to pay Rs 330000 for the plot, find the C.P. of the plot for Yogita.

10. A dealer purchased 250 bulbs for Rs 10 each. However, 12 bulbs got fused which she had to throw away. She sold the remaining bulbs at Rs 12 each. Find her gain or loss per cent in the transaction.

11. The difference between two selling prices of a shirt at profits of 4 % and 5 % is Rs 6. Find: (i) C.P. of the shirt (ii) the two selling prices of the shirt.

12. Toshiba bought 100 hens for Rs 8000 and sold 20 of these at a gain of 5 %. At what gain per cent must she sell the remaining hens so as to gain 20 % on the whole?

13. Some toffees are bought at the rate of 11 for Rs 10 and the same number at the rate of 9 for Rs 10. If the whole lot is sold at one rupee per toffee, find the gain or loss per cent on the whole transaction.

    [*Hint* : Let the number of toffees bought be the L.C.M. of 11 and 9.]

14. Shabana bought 16 dozen ball pens and sold them at a loss equal to S.P. of 8 ball pens. Find

    (i) her loss per cent,

    (ii) S.P. of 1 dozen ball pens, if she purchased these 16 dozen ball pens for Rs 576.

15. Mariam bought two fans for Rs 3605. She sold one at a profit of 15 % and the other at a loss of 9 %. If Mariam obtained the same amount for each fan, find the cost price of each fan.

    [*Hint* : If C.P. of the first fan = Rs x, C.P. of the second fan = Rs $(3605 - x)$]

4.3 Discount

Though buying and selling appears to be a simple affair, yet it is not so in reality. Shopkeepers want to get the maximum price for their goods while the customers want to pay as little as possible for the same goods. Shopkeepers devise several schemes to attract customers. Selling things at credit is one such scheme. However, in reality, shopkeepers are more interested in getting cash money for their goods. They often offer a certain type of rebate

or incentive to the customers. Incentive is offered for making customers believe that they are getting goods at cheaper rates. Shopkeepers offer incentives to the customers in various forms. For example, you may come across advertisements of the following types:

'Now get 1100 g detergent for the cost of just 1 kg'

'Get a mug free with every 500 g pack of tea'

'10 % festival discount on all bed sheets'

or 'Rs 2 off on the price of your favourite toilet soap' and so on.

Thus, sometimes incentive is also provided by reducing the price attached to the article which the shopkeeper professes to be the cost of the article for the customer. This attached price is called the *marked price* (M.P.) or *list price* (L.P.) of the article and the reduction allowed in the price is called the *discount*. It is generally given as a certain per cent of the marked price. Customer pays the difference between the marked price and the discount. In other words (*marked price – discount*) is what the customer actually pays. This amount is, therefore, the selling price (S.P.) of the article.

Thus, $\text{Discount} = \text{M.P.} - \text{S.P.}$ (1)

Further, $\text{Rate of Discount} = \text{Discount }\% = \dfrac{\text{Discount}}{\text{M.P.}} \times 100$ (2)

Now, $\text{S.P.} = \text{M.P.} - \text{Discount}$ [From (1)]

$\therefore$ $\text{S.P.} = \text{M.P.} - \dfrac{\text{Discount }\% \times \text{M.P.}}{100}$ [From (2)]

or $\text{S.P.} = \text{M.P.} \times \left(1 - \dfrac{\text{Discount }\%}{100}\right)$

or $\text{S.P.} = \text{M.P.} \times \left(\dfrac{100 - \text{Discount }\%}{100}\right)$ (3)

From (3), we can easily write

$$\text{M.P.} = \frac{100 \times \text{S.P.}}{100 - \text{Discount }\%} \quad (4)$$

Let us now take some examples to illustrate the above ideas.

Example 5 : Marked price of a book is Rs 30. It is sold at a discount of 15 %. Find the discount allowed on the book and its selling price.

Solution : M.P. of the book = Rs 30

Rate of discount = 15 %

$\therefore$ Discount allowed = 15 % of Rs 30

$$= \frac{15}{100} \times \text{Rs } 30 = \text{Rs } 4.50$$

Therefore, S.P. of the book = Rs 30 – Rs 4.50 = Rs 25.50.

Example 6 : A table with marked price Rs 1200 was sold to a customer for Rs 1100. Find the rate of discount allowed on the table.

Solution : M.P. = Rs 1200

S.P. = Rs 1100

$\therefore$ Discount = Rs 1200 – Rs 1100 = Rs 100

$$\text{Rate of discount} = \frac{\text{Discount}}{\text{M. P.}} \times 100\ \%$$

$$= \frac{100}{1200} \times 100\,\% = 8\frac{1}{3}\ \%$$

Example 7 : A shirt was sold for Rs 442 after allowing a discount of 15 % on the marked price. Find the marked price of the shirt.

Solution : Let M.P. be Rs x.

$\therefore$ Discount = 15% of Rs x

$$= \text{Rs } \frac{15}{100} \times x \ = \text{Rs } \frac{3x}{20}$$

$$\therefore \quad \text{S.P.} = \text{Rs}\left(x - \frac{3x}{20}\right) = \text{Rs } \frac{17x}{20}$$

According to the given condition,

$$\frac{17x}{20} = 442$$

or $$x = \frac{442 \times 20}{17} = 520$$

Thus, marked price of the shirt is Rs 520.

Example 8 : A shopkeeper offers 10 % off-season discount to the customers and still makes a profit of 26 %. What is the cost price for the shopkeeper of a pair of shoes marked at Rs 1120 ?

Solution : M.P. = Rs 1120

Rate of discount = 10 %

$\therefore$ Discount allowed = Rs $\frac{10}{100} \times 1120 =$ Rs 112

Thus, S.P. of the pair of shoes = Rs (1120 – 112) = Rs 1008

Now profit % of the shopkeeper = 26

Therefore, $$\text{C.P.} = \frac{100 \times \text{S.P.}}{100 + \text{Profit}\,\%}$$

$$= \text{Rs } \frac{100 \times 1008}{100 + 26} = \text{Rs } \frac{100 \times 1008}{126} = \text{Rs } 800$$

Thus, the cost price of the pair of shoes is Rs 800.

Example 9 : Jyoti and Meena run a readymade garments shop. They mark the garments at such a price that even after allowing a discount of 12.5 %, they make a profit of 10 %. Find the marked price of a suit which costs them Rs 1470.

Solution : C.P. of a suit = Rs 1470

Profit = 10 % of Rs 1470

$$= \text{Rs } \frac{10}{100} \times 1470 = \text{Rs } 147$$

$\therefore$ S.P. of the suit = Rs (1470 + 147)

= Rs 1617

Let the marked price be Rs 100. Then,

Discount = 12.5 % of Rs 100

= Rs 12.50

$\therefore$ S.P. = Rs (100 – 12.50) = Rs 87.50

Now if S.P. is Rs 87.50, M.P. = Rs 100

$\therefore$ If S.P. is Rs 1617, M.P. = Rs $\frac{100}{87.50} \times 1617$

= Rs 1848

Thus, the marked price of the suit is Rs 1848.

Alternate Method for M.P. : Let the marked price be Rs x.
We have

$$\text{M.P.} = \frac{100 \times \text{S.P.}}{100 - \text{Discount}\,\%} \qquad \text{[From (4)]}$$

or

$$x = \frac{100 \times 1617}{(100 - 12.5)} = \frac{161700}{87.5} = 1848$$

Thus, the marked price of the suit is Rs 1848.

EXERCISE 4.2

1. Find the S.P. when
   (i) M.P. = Rs 320 and discount = 12.5 %.
   (ii) M.P. = Rs 990 and discount = 10 %.
2. Find the rate of discount when
   (i) M.P. = Rs 250 and S.P. = Rs 235
   (ii) M.P. = Rs 1880 and S.P. = Rs 1504
3. Find the M.P. when
   (i) S.P. = Rs 1920 and discount = 4 %.
   (ii) S.P. = Rs 2970 and discount = 1%.
4. A discount of 3 % is offered on the marked price of sewing machines. What cash amount will a customer pay for a sewing machine, the price of which is marked at Rs 1300?
5. List price of a scooter is Rs 35000. It is available at a discount of 8 %. Find the selling price of the scooter.
6. A table of marked price Rs 1500 is sold for Rs 1080 after allowing a certain discount. Find the rate of discount.
7. Find the marked price of an almirah which is sold at Rs 5225 after allowing a discount of 5%.
8. Chandu purchased a watch at 20 % discount on its marked price but sold it at the marked price. Find the gain per cent of Chandu on this transaction.
9. A lady shopkeeper allows her customers 10 % discount on the marked price of the goods and still gets a profit of 25 %. What is the cost price of a fan for her marked at Rs 1250?
10. Jasmine allows 4 % discount on the marked price of her goods and still earns a profit of 20 %. What is the cost price of a shirt for her marked at Rs 850 ?

11. What price should a trader mark on an article that costs him Rs 918 to gain 20 % after allowing a discount of 15%?

12. What price should Suneeta mark on a sari which costs her Rs 2200 so as to gain 26 % after allowing a discount of 12 %?

13. A cycle merchant allows 25 % discount on the marked price of the cycles and still makes a profit of 20 %. If he gains Rs 360 over the sale of one cycle, find the marked price of the cycle.

[*Hint* : First find the C.P. of a cycle.]

14. What price should Aslam mark on a pair of shoes which costs him Rs 1200 so as to gain 12 % after allowing a discount of 16 % ?

15. A dealer of scientific instruments allows 20 % discount on the marked price of the instruments and still makes a profit of 25 %. If his gain over the sale of a instrument is Rs 150, find the marked price of the instrument.

Things to Remember

1. In case of profit,

$$\text{S.P.} = \text{C.P.} \times \left(\frac{100 + \text{Profit}\,\%}{100}\right)$$

$$\text{C.P.} = \frac{100 \times \text{S.P.}}{100 + \text{Profit}\,\%}$$

2. In case of loss,

$$\text{S.P.} = \text{C.P.} \times \frac{(100 - \text{Loss}\,\%)}{100}$$

$$\text{C.P.} = \frac{100 \times \text{S.P.}}{100 - \text{Loss}\,\%}$$

3. Discount is usually expressed as a certain per cent of the M.P.

4. Discount = M.P. – S.P.

5. Rate of Discount = Discount % = $\dfrac{\text{Discount}}{\text{M.P.}} \times 100$

6. $\text{S.P.} = \text{M.P.} \times \left(\dfrac{100 - \text{Discount}\,\%}{100}\right)$

7. $\text{M.P.} = \dfrac{100 \times \text{S.P.}}{100 - \text{Discount}\,\%}$

CHAPTER

5

COMPOUND INTEREST

5.1 Introduction

In Class VII, you have learnt about simple interest. Recall that in the case of simple interest, the interest is charged only on the original sum borrowed (principal) throughout the period of loan. However, in dealings of day-to-day life the interest that is charged/paid is rarely the simple interest. The interest which the banks, post offices, insurance corporations and other finance companies charge/pay is not the simple interest. In these cases, the interest, as it falls due over a period of time, is again invested to earn further interest. Thus, the interest is added to the principal to form a *new* principal which may be reinvested for the next time period. This process can be repeated for several time periods. The difference between the original principal and the amount at the end of the last time period is known as the *compound interest* on the original principal for that period.

The time period after which interest is added each time to form a new principal is called the *conversion period*. It may be one year, six months, three months or one month and in these cases, the interest is said to be *compounded* annually, semi-annually, quarterly or monthly, respectively.

In this Chapter, we shall discuss the concept of compound interest and the methods of calculating the amount and the compound interest. Further, we shall take up only those cases in which interest is *compounded annually*.

5.2 Compound Interest

To understand the concept of compound interest, we first take up an example of simple interest. Let us suppose that a sum of Rs 5000 is borrowed at 8 % simple interest for 2 years. What will be the total interest on this sum?

We know that

$$\text{Simple Interest} = \frac{P \times R \times T}{100},$$

where P is the principal, R (per cent) is the rate of interest per year and T is the number of years. Thus,

$$\text{Simple Interest} = \text{Rs } \frac{5000 \times 8 \times 2}{100}$$
$$= \text{Rs } 800$$

Let us now consider the following situation :

Harsh borrows a sum of Rs 5000 from a finance company for one year that charges interest at the rate of 8 % per annum. Then at the end of one year, Harsh owes the company :

(i) borrowed sum (principal) = Rs 5000

and (ii) interest on Rs 5000 for one year at the rate of 8 % per annum

$$= \text{Rs } \frac{5000 \times 8 \times 1}{100}$$
$$= \text{Rs } 400$$

Thus, Harsh owes to the company = Rs 5000 + Rs 400
= Rs 5400

Suppose somehow, Harsh is not in a position to pay this amount to the company. Obviously, the company will charge from him the interest on Rs 5400 thereafter. Thus, the principal for the second year would be Rs 5400, which is the amount at the end of the first year.

At the end of the second year, Harsh owes the company :

(i) new principal = Rs 5400

and (ii) interest on the new principal for one year at the rate of 8 % per annum

$$= \text{Rs } \frac{5400 \times 8 \times 1}{100}$$
$$= \text{Rs } 432$$

Thus, Harsh now owes the company the following sum :

Rs 5400 + Rs 432, i.e., Rs 5832

Hence, total interest payable to the company = Rs 5832 – Rs 5000
= Rs 832

The interest calculated in this manner is called *compound interest*. In the case above, the interest Rs 832 is the compound interest on Rs 5000 for 2 years at the rate of 8 % per annum.

Note that the simple interest on Rs 5000 for the same period (2 years) and at the same rate of interest (8 %) was Rs 800 whereas the compound interest is Rs 832, i.e., Rs (832 – 800) or Rs 32 more than the simple interest. The difference is due to the fact that in the case of compound interest, we had added Rs 400, the simple interest of the first year, to the principal Rs 5000 and treated Rs (5000 + 400), i.e., Rs 5400 as the *new principal* for the second year. In other words, in case of compound interest, during the second year, interest was charged on the interest of the first year also.

You may verify that the difference of Rs 32 between the compound interest and the simple interest is the interest on Rs 400 (the interest of the first year) for one year.

Remarks : 1. The main difference between simple interest and compound interest is that in case of simple interest, the principal remains constant throughout, whereas in the case of compound interest, it goes on changing periodically.

2. In case of compound interest, the principal for the second year is the sum of the principal for the first year and the simple interest for the first year. Similarly, the principal for the third year is the sum of the principal for the second year and the simple interest for the second year and so on.
3. For a given principal, rate and time, the compound interest is generally, more than the simple interest. For the first year, simple interest and compound interest are the same (provided the interests are calculated yearly).
4. In the above example, the interest was calculated on yearly basis. As mentioned earlier, this is referred to as saying that the interest is compounded annually.
5. Similar procedure can be applied to compute the compound interest on a given sum for more than 2 years.

Let us consider some examples :

Example 1 : Find the compound interest on Rs 8000 for 2 years at 6 % per annum.

Solution : Let us first find out the compound interest on Rs 100 for 2 years at 6 % per annum.

Interest on Rs 100 for one year at 6 % per annum = Rs 6

Thus, for the first year,

Principal = Rs 100

Interest = Rs 6

$\therefore$ Amount = Rs 106

This amount becomes the new principal for the second year.

For the second year,

$$\text{Principal} = \text{Rs } 106$$

$$\text{Interest} = \text{Rs } \frac{106 \times 6 \times 1}{100} = \text{Rs } 6.36$$

$$\therefore \quad \text{Amount} = \text{Rs } 106 + \text{Rs } 6.36 = \text{Rs } 112.36$$

Thus, Compound Interest $= \text{Rs } 112.36 - \text{Rs } 100$

$= \text{Rs } 12.36$

Now, Compound Interest on Rs 100 = Rs 12.36

$$\therefore \quad \text{Compound Interest on Re } 1 = \text{Rs } \frac{12.36}{100}$$

Hence, Compound Interest on Rs 8000 $= \text{Rs } \frac{12.36 \times 8000}{100}$

$= \text{Rs } 988.80$

Example 2 : Find the amount and the compound interest on Rs 20000 for 3 years at 10 % per annum.

Solution : We first find out the compound interest on Rs 100 for 3 years at 10 % per annum.

Interest on Rs 100 at 10 % for 1 year = Rs 10

Thus, amount at the end of the first year $= \text{Rs } (100 + 10)$

$= \text{Rs } 110.$

This forms the principal for the second year.

$$\text{Interest for the second year} = \text{Rs } \left(\frac{110 \times 10 \times 1}{100}\right)$$

$= \text{Rs } 11$

$\therefore$ Amount at the end of the second year = Rs 110 + Rs 11

$= \text{Rs } 121$

Again, this forms the principal for the third year.

$$\therefore \quad \text{Interest for the third year} = \text{Rs } \left(\frac{121 \times 10 \times 1}{100}\right)$$

$= \text{Rs } 12.10$

$\therefore$ Amount at the end of the third year = Rs 121 + Rs 12.10

$= \text{Rs } 133.10$

$\therefore$ Compound Interest = Rs 133.10 – Rs 100

= Rs 33.10

Now, Amount on Rs 100 = Rs 133.10

$\therefore$ Amount on Re 1 = Rs $\frac{133.10}{100}$

Hence, Amount on Rs 20000 = Rs $\frac{133.10}{100} \times 20000$

= Rs 26620

Compound Interest = Amount – Principal

= Rs 26620 – Rs 20000

= Rs 6620

Thus, the required Amount is Rs 26620 and the compound interest is Rs 6620.

Observe that in the above examples, we have first calculated Amount/ Compound Interest on Rs 100 and then applied the Unitary Method to calculate Amount/ Compound Interest on the given principal. We can also calculate Amount /Compound Interest directly for the given sum as shown in the next example.

Example 3 : Surbhi borrowed a sum of Rs 12000 from a finance company to purchase a refrigerator. If the rate of interest is 5 % per annum compounded annually, calculate the compound interest that Surbhi has to pay to the company after three years.

Solution : Principal for the first year = Rs 12000

Interest for the first year = Rs $\frac{12000 \times 5 \times 1}{100}$ = Rs 600

Amount at the end of first year = Rs 12000 + Rs 600

= Rs 12600,

which is the principal for the second year

Interest for the second year = Rs $\left(\frac{12600 \times 5 \times 1}{100}\right)$

= Rs 630

Amount at the end of the second year = Rs 12600 + Rs 630 = Rs 13230, which is the principal for the third year.

Interest for the third year = Rs $\left(\frac{13230 \times 5 \times 1}{100}\right)$

= Rs 661.50

Amount at the end of the third year = Rs 13230 + Rs 661.50

= Rs 13891.50

∴ Compound interest after three years

= Amount at the end of the third year – Original principal

= Rs 13891.50 – Rs 12000

= Rs 1891.50

Note : Compound Interest after three years can also be calculated as the sum of interests for the first, second and third years (Rs 600 + Rs 630 + Rs 661.50).

EXERCISE 5.1

1. Compute the compound interest on :

(i) Rs 1500 for 2 years at 6 % per annum.

(ii) Rs 2860 for 2 years at 5 % per annum.

(iii) Rs 3000 for 2 years at 5 % per annum.

(iv) Rs 5000 for 2 years at 10 % per annum.

(v) Rs 8500 for 2 years at 8 % per annum.

2. Find the compound interest on Rs 8000 for 2 years at $12\frac{1}{2}$ % per annum.

3. Find the compound interest on Rs 10000 for 3 years at 10 % per annum compounded annually.

4. Salma deposited Rs 6250 with a Company at 9.5 % per annum compound interest for 2 years. Calculate the amount she will get after 2 years.

5. Find the compound interest on Rs 28000 for 3 years at 5 % per annum compounded annually.

6. Find the amount and compound interest on a sum of Rs 15625 at 4 % per annum for 3 years compounded annually.

7. Vasudevan invested Rs 8000 at an interest rate of 9 % per annum. Find the total amount he will get after 3 years, if the interest is compounded annually.

8. Daljit received a sum of Rs 40000 as a loan from a finance company. If the rate of interest is 7 % per annum compounded annually, calculate the compound interest that Daljit pays after 2 years.

9. To renovate his shop, Arif obtained a loan of Rs 8000 from a bank. If the rate of interest is 5 % per annum compounded annually, calculate the compound interest that Arif will have to pay after 3 years.

10. Anil borrowed a sum of Rs 9600 to install a handpump in his dairy. If the rate of interest is $5\frac{1}{2}$ % per annum compounded annually, determine the compound interest which Anil will have to pay after 3 years.

11. Maria invests Rs 93750 at 9.6 % per annum for 3 years and the interest is compounded annually. Calculate

(i) the amount standing to her credit at the end of the second year,

(ii) the interest for the 3rd year.

5.3 Formula for Finding the Compound Interest

Let us go back to Examples 1 and 2 discussed in the previous section. You can see that as the period of time increases, the process of computing the interest becomes more lengthy and time consuming. We should, therefore, try to find a formula for determining the compound interest.

In Example 1, we can also perform the calculation of amount as follows :

$$\text{Interest for first year} = \text{Rs}\left(\frac{8000 \times 6 \times 1}{100}\right)$$

$$\therefore \quad \text{Amount at the end of the first year} = \text{Rs } 8000 + \text{Rs}\left(\frac{8000 \times 6 \times 1}{100}\right)$$

$$= \text{Rs } 8000\left(1 + \frac{6}{100}\right) \qquad (1)$$

But the amount (1) forms the principal for the second year.

$$\therefore \quad \text{Interest for the second year} = \text{Rs } 8000\left(1 + \frac{6}{100}\right) \times \frac{6 \times 1}{100}$$

$\therefore$ Amount at the end of the second year

$$= \text{Rs}\left[8000\left(1 + \frac{6}{100}\right) + 8000\left(1 + \frac{6}{100}\right) \times \frac{6}{100}\right]$$

$$= \text{Rs } 8000\left(1 + \frac{6}{100}\right)\left(1 + \frac{6}{100}\right)$$

$$= \text{Rs } 8000\left(1 + \frac{6}{100}\right)^2 \qquad (2)$$

In Example 2, we can also perform the calculation of amount as follows :

$$\text{Interest for the first year} = \text{Rs}\left(\frac{20000 \times 10 \times 1}{100}\right)$$

$\therefore$ Amount at the end of the first year $= \text{Rs}\left(20000 + \frac{20000 \times 10 \times 1}{100}\right)$

$$= \text{Rs } 20000\left(1 + \frac{10}{100}\right) \qquad (3)$$

But the amount (3) forms the principal for the second year.

$\therefore$ Interest for the second year $= \text{Rs } 20000\left(1 + \frac{10}{100}\right) \times \frac{10 \times 1}{100}$

$\therefore$ Amount at the end of the second year

$$= \text{Rs}\left[20000\left(1 + \frac{10}{100}\right) + 20000 \times \left(1 + \frac{10}{100}\right) \times \frac{10}{100}\right]$$

$$= \text{Rs } 20000\left(1 + \frac{10}{100}\right)\left(1 + \frac{10}{100}\right)$$

$$= \text{Rs } 20000\left(1 + \frac{10}{100}\right)^2 \qquad (4)$$

But the amount (4) forms the principal for the third year.

$\therefore$ Interest for the third year $= \text{Rs } 20000\left(1 + \frac{10}{100}\right)^2 \times \frac{10 \times 1}{100}$,

and the amount at the end of the third year

$$= \text{Rs}\left[20000\left(1 + \frac{10}{100}\right)^2 + 20000\left(1 + \frac{10}{100}\right)^2 \times \frac{10}{100}\right]$$

$$= \text{Rs } 20000\left(1 + \frac{10}{100}\right)^2\left(1 + \frac{10}{100}\right)$$

$$= \text{Rs } 20000\left(1 + \frac{10}{100}\right)^3 \qquad (5)$$

Note that :

In Example 1:

The rate of interest = 6 %, the number of years = 2, Principal = Rs 8000

and the amount $A = \text{Rs } 8000\left(1+\frac{6}{100}\right)^2$ [From (2) above]

In Example 2 :

The rate of interest = 10 %, the number of years = 3, Principal = Rs 20000

and the amount $A = \text{Rs } 20000\left(1+\frac{10}{100}\right)^3$ [From (5) above]

From the above discussion, we may say that

$$A = P\left(1+\frac{r}{100}\right)^n, \qquad \text{(I)}$$

where A is the amount after the last time-period (i.e. year), P the principal, r the rate per cent per year and n is the number of years.

Formula (I) gives the *amount* at compound interest. Now, compound interest can be determined as follows :

Compound interest = A – P

$$= P\left(1+\frac{r}{100}\right)^n - P \qquad \text{[Using (I)]}$$

$$= P\left[\left(1+\frac{r}{100}\right)^n - 1\right] \qquad \text{(II)}$$

To illustrate the use of these formulae, we consider some more examples.

Example 4 : Find the amount of Rs 6000 lent at 5 % per annum for 2 years if the interest is compounded annually.

Solution : Here P = Rs 6000, $r = 5$, $n = 2$.

Using Formula (I) for calculating the amount, we have

$$A = P\left(1+\frac{r}{100}\right)^n = \text{Rs } 6000\left(1+\frac{5}{100}\right)^2$$

$$= \text{Rs } 6000\left(\frac{105}{100}\times\frac{105}{100}\right)$$

$$= \text{Rs}\left(6000\times\frac{21}{20}\times\frac{21}{20}\right)$$

$$= \text{Rs } 6615$$

Therefore, the required amount is Rs 6615.

In the above Example, we have explicitly mentioned that interest is compounded annually. However, if nothing is mentioned, it is understood that the interest is compounded annually.

Example 5 : Find the compound interest on Rs 25600 for 2 years at 6.25 % per annum.

Solution : We have

$$P = \text{Rs } 25600,\ n = 2,\ r = 6.25$$

$$\therefore \quad \text{Compound interest} = P\left[\left(1+\frac{r}{100}\right)^n - 1\right] \qquad \text{[By Formula (II)]}$$

$$= \text{Rs}\left[25600\left\{\left(1+\frac{6.25}{100}\right)^2 - 1\right\}\right]$$

$$= \text{Rs}\left[25600\left\{\left(1+\frac{1}{16}\right)^2 - 1\right\}\right]$$

$$= \text{Rs}\left[25600\left\{\left(\frac{17}{16}\right)\left(\frac{17}{16}\right) - 1\right\}\right]$$

$$= \text{Rs}\left[25600\left(\frac{289-256}{256}\right)\right]$$

$$= \text{Rs } \frac{25600 \times 33}{256} = \text{Rs } 3300$$

Example 6 : Find the amount and compound interest on Rs 16000 for 3 years, if the rate of interest is 2 % per annum.

Solution : Here P = Rs 16000, $n = 3$, $r = 2$.

$$\therefore \quad \text{Amount} = \text{Rs } 16000\left(1+\frac{2}{100}\right)^3 \qquad \text{[By Formula (I)]}$$

$$= \text{Rs } 16000\left(\frac{102}{100}\right)\left(\frac{102}{100}\right)\left(\frac{102}{100}\right)$$

$$= \text{Rs } 16000\left(\frac{51}{50}\right)\left(\frac{51}{50}\right)\left(\frac{51}{50}\right)$$

$$= \text{Rs } 16979.33 \text{ approximately}$$

and compound interest = Rs (16979.33 – 16000) approximately
= Rs 979.33

Example 7 : Simple interest on a sum of money for 3 years at $6\frac{1}{4}$ % per annum is Rs 2400. What will be the compound interest on that sum at the same rate for the same period?

Solution : Let the sum of money be Rs P. Then,

$$\frac{P \times \frac{25}{4} \times 3}{100} = 2400 \qquad \left[\begin{array}{l}\text{Using the Formula}\\ \text{S.I.} = \dfrac{P \times R \times T}{100}\end{array}\right]$$

or
$$P = \frac{240000 \times 4}{75}$$
$$= 12800$$

Compound Interest on Rs 12800 for 3 years at $6\frac{1}{4}$ %

$$= \text{Rs}\left[12800\left\{\left(1+\frac{25}{400}\right)^3 - 1\right\}\right]$$

$$= \text{Rs}\left[12800\left\{\left(1+\frac{1}{16}\right)^3 - 1\right\}\right]$$

$$= \text{Rs}\left[12800\left\{\frac{17}{16} \times \frac{17}{16} \times \frac{17}{16} - 1\right\}\right]$$

$$= \text{Rs } 12800\left\{\frac{4913}{4096} - 1\right\}$$

$$= \text{Rs } 12800\left(\frac{817}{4096}\right)$$

= Rs 2553.13 approximately

Example 8 : The difference between the compound interest and the simple interest on a certain sum for 2 years at 10 % per annum is Rs 300. Find the sum.

Solution : Let the sum be Rs P.

Simple interest on Rs P for 2 years at 10 % per annum

$$= \text{Rs } \frac{P \times 10 \times 2}{100} = \text{Rs } \frac{P}{5} \qquad (1)$$

Compound Interest on Rs P for 2 years at 10 % per annum

$$= \text{Rs P}\left[\left(1+\frac{10}{100}\right)^2 - 1\right]$$

$$= \text{Rs P}\left[\frac{11}{10}\times\frac{11}{10} - 1\right]$$

$$= \text{Rs P}\left(\frac{121-100}{100}\right)$$

$$= \text{Rs}\ \frac{21\text{P}}{100} \qquad (2)$$

Given that compound interest – simple interest = Rs 300, we find from (1) and (2) that

$$\frac{21\text{P}}{100} - \frac{\text{P}}{5} = 300$$

or $\quad 21\text{P} - 20\text{P} = 30000$

or $\quad \text{P} = 30000$

Hence, the required sum is Rs 30000.

Example 9 : A certain sum amounts to Rs 5832 in 2 years at 8 % compound interest. Find the sum.

Solution : Here A = Rs 5832, $r = 8$, $n = 2$.

Let P be the required sum in rupees. Then

$$5832 = \text{P}\left(1+\frac{8}{100}\right)^2$$

or $$5832 = \text{P}\left(\frac{108}{100}\right)^2$$

or $$5832 = \text{P}\left(\frac{27}{25}\right)\left(\frac{27}{25}\right)$$

$$\therefore \quad \text{P} = \frac{5832\times 25\times 25}{27\times 27}$$

$$= 5000$$

Hence, the required sum is Rs 5000.

Example 10 : At what rate per cent will a sum of Rs 1000 amount to Rs 1102.50 in 2 years at compound interest?

Solution : Let r be the rate per cent per annum. Here,

$$A = \text{Rs } 1102.50,\ P = \text{Rs } 1000,\ n = 2.$$

We have

$$1102.50 = 1000\left(1+\frac{r}{100}\right)^2 \qquad \text{[Using Formula (I)]}$$

or $$\left(1+\frac{r}{100}\right)^2 = \frac{1102.50}{1000} = \frac{441}{400} = \left(\frac{21}{20}\right)^2$$

$\therefore$ $$1+\frac{r}{100} = \frac{21}{20}$$

or $$\frac{r}{100} = \frac{21}{20} - 1 = \frac{1}{20}$$

$\therefore$ $$r = 5$$

Hence, the rate of interest is 5 % per annum.

Example 11 : The compound interest on Rs 1800 at 10 % per annum, for a certain period of time is Rs 378. Find the time in years.

Solution : Principal = Rs 1800

Compound Interest = Rs 378

Let the time be n years. Then

Amount = Rs (1800 + 378) = Rs 2178.

Now $$2178 = 1800\left(1+\frac{10}{100}\right)^n$$

or $$\left(1+\frac{10}{100}\right)^n = \frac{2178}{1800} = \frac{121}{100} = \left(\frac{11}{10}\right)^2$$

or $$\left(\frac{11}{10}\right)^n = \left(\frac{11}{10}\right)^2$$

or $$n = 2$$

Therefore, the given sum of Rs 1800 will yield a compound interest of Rs 378 at 10 % in 2 years.

EXERCISE 5.2

In Questions 1 to 5, calculate the amount and the compound interest by using the formulae for compound interest.

1. Principal = Rs 4000, Rate = 5 % per annum, Time = 2 years
2. Principal = Rs 6000, Rate = 10 % per annum, Time = 2 years
3. Principal = Rs 6250, Rate = 4 % per annum, Time = 2 years
4. Principal = Rs 20000, Rate = 7.5 % per annum, Time = 3 years
5. Principal = Rs 31250, Rate = 8 % per annum, Time = 3 years
6. Find the compound interest on Rs 25000 for 3 years at 6 % per annum.
7. Find the amount of Rs 125000 after 3 years, when the interest is compounded annually at the rate of 6 % per annum.
8. Hema lent Rs 40960 to Hameed at the rate of $6\frac{1}{4}$ % per annum compound interest. Find the amount payable by Hameed to Hema after 3 years.
9. Chandran borrowed a sum of Rs 10000 from Varsha for 3 years. If the rate of interest is 10 % per annum, compounded annually, find the amount that Chandran will have to pay at the end of 3 years.
10. Find the difference between the compound interest and the simple interest on Rs 32000 at 12 % per annum for 3 years.
11. Fatima borrows Rs 12500 at 12 % per annum for 3 years at simple interest and Radha borrows the same sum for the same time at 10 % per annum, compounded annually, Who pays more interest and by how much?
12. I borrow Rs 12000 from Jamshed at 6 % per annum simple interest for 2 years. Had I borrowed this sum at 6 % per annum compound interest, what excess amount would I have to pay to him?
13. Simple interest on a sum of money for 2 years at $6\frac{1}{2}$ % per annum is Rs 5200. What will be the compound interest on that sum at the same rate for the same time period?
14. Find the compound interest at the rate of 5 % per annum for 3 years on that principal which in 3 years at the rate of 5 % per annum gives Rs 1200 as simple interest.
15. The difference between the compound interest and simple interest on a certain sum for 2 years at 7.5 % per annum is Rs 360. Find the sum.
16. The difference in simple interest and compound interest on a certain sum of money at $6\frac{2}{3}$ % per annum for 3 years is Rs 46. Determine the sum.

17. A sum of money deposited at 2% per annum compounded annually becomes Rs 10404 at the end of 2 years. Find the sum deposited.

18. A sum of money amounts to Rs 453690 in 2 years at 6.5 % per annum compounded annually. Find the sum.

19. What sum of money will amount to Rs 45582.25 at $6\frac{3}{4}$% per annum in two years, interest being compounded annually?

20. At what rate per cent per annum will a sum of Rs 4000 yield compound interest of Rs 410 in 2 years?

21. In how much time will a sum of Rs 1600 amount to Rs 1852.20 at 5 % per annum compound interest?

22. Rekha invested a sum of Rs 12000 at 5 % per annum compound interest. She received an amount of Rs 13230 after *n* years. Find the value of *n*.

Things to Remember

1. If the principal remains the same throughout the loan period, then the interest calculated on this principal is called the simple interest.
2. The time period after which interest is added each time to form a new principal is called the conversion period and the interest so obtained is called the compound interest.
3. If the conversion period is one year, the interest is said to be compounded annually.
4. The main difference between the simple interest and the compound interest on a certain sum is that in the case of simple interest, the principal remains constant throughout whereas in the case of compound interest, it goes on changing periodically.
5. $$A = P\left(1 + \frac{r}{100}\right)^n$$

   $$\text{Compound Interest} = A - P = P\left[\left(1 + \frac{r}{100}\right)^n - 1\right],$$

   where A is the amount, P the principal, *r* the rate per cent per conversion period and *n* is the number of conversion periods.

As History Tells Us

The history of Commercial Mathematics is as old as human civilization. During the 19th century, half a million clay tablets were excavated in Mesopotamia. Three hundred of these tablets relate to Mathematics alone. It is through these tablets mainly that we have come to know about the commercial activities of the oldest known human civilization of the Babylonians (including Sumerians).

About two hundred of these tablets are purely *table*–tablets giving tables of multiplications, reciprocals, squares, cubes and exponentials. In particular, we find tablets giving tables for a^n for $n = 1, 2, \ldots, 10$ and $a = 9, 16, \ldots, 100, \ldots, 225$. There is evidence that these tablets were used for calculating simple and compound interests. On a tablet (now in the Museum at *Louvre* in France) of period dating about 1700 B.C., this problem (stated here in current notation) is given : *Find how long will it take for a certain sum of money to double itself at compound interest of* 20 %. To solve this, you have to find n so that $P(1+\frac{20}{100})^n = 2P$ or $(1.2)^n = 2$. It was computed from the tablet that $(1.2)^3$, i.e. $1.728 < 2$ and $(1.2)^4$, i.e., $2.0736 > 2$. They, then interpolated the value of n between 3 and 4 (i.e., solved the problem that if 1.728 and 2.0736 correspond, respectively to 3 and 4 then what value will correspond to 2.)

The concepts of simple and compound interest are found in almost all the later civilizations. Greeks, Romans, Italians, British, Jews all dealt with interest. In all the books on Arithmetic of 16th and 17th centuries the concept was given prominence. Many of these give tables for calculation of compound interest.

In India too, the concept of interest is known since the Sutra period, a few centuries B.C. *Mahavira* (850) and *Bhaskara* (1150) give several problems on interest that indicate that here the interest was calculated on per cent basis.

The topic of Profit and Loss also has been dealt with in all the books on Arithmetic of the 16th centuries onwards. The phrase was coined in Italy and through successive translations in Latin, German, Dutch and French, it reached English authors as Loss and Gain.

The concept of *Discount* has its origin in *commissions* and *brokerage* which were charged by a middleman in sale and purchase of goods. Discount as (i) *reduction* in *payment* if the money due at a certain date is paid earlier and (ii) reduction in the marked price as an incentive for purchase is relatively a recent feature.

CHAPTER

6

ALGEBRAIC IDENTITIES

6.1 Introduction

In Class VII, you studied some algebraic identities. As you know, an algebraic identity is an algebraic relation of equality that remains true for all values of the literals occuring in the relation. You studied the following algebraic identities :

A. $(a + b)^2 = a^2 + 2ab + b^2$

B. $(a - b)^2 = a^2 - 2ab + b^2$

C. $(a + b)(a - b) = a^2 - b^2$

Recall that to find the factors of an algebraic expression, we write it as a product of algebraic expressions. Each expression in the product is called a *factor*. The process of finding factors is called *factorisation*.

In this Chapter, we shall learn some more identities. We shall also learn the use of these identities in factorising algebraic expressions.

6.2 Some More Standard Identities

I. The Product $(x + a)(x + b)$

We can multiply two identical binomials using the Identity A above. Let us now find the product when the second terms of the two binomials are different. Multiplying the binomials $x + a$ and $x + b$,

$$
\begin{aligned}
(x + a)(x + b) &= x(x + b) + a(x + b) \\
&= x^2 + xb + ax + ab \\
&= x^2 + bx + ax + ab && \text{(Since } xb = bx\text{)} \\
&= x^2 + ax + bx + ab && \text{(Since } bx + ax = ax + bx\text{)} \\
&= x^2 + (a + b)x + ab && \text{(Taking } x \text{ as a common factor from } ax \text{ and } bx\text{)}
\end{aligned}
$$

Thus, we have the following identity :

Identity I :

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

or

$$(x + a)(x + b) = x^2 + ax + bx + ab$$

Activity 1 : Recall that the product of the two binomials $x + a$ and $x + b$ may be regarded as the area of a rectangle with sides $x + a$ and $x + b$. On a piece of cardboard (or an old greeting card), draw a rectangle ABCD with sides $x + a$ and $x + b$ [Fig.6.1 (i)]. Clearly, the area of rectangle ABCD is $(x + a)(x + b)$.

Mark a point P on AB let AP = x [Fig. 6.1 (ii)]. Mark a point Q on AD and let AQ = x [Fig.6.1(ii)]. Since AB = $x + a$ and AP = x, therefore, PB = a. Again since AD = $x + b$ and AQ = x, therefore, QD = b [Fig.6.1(ii)].

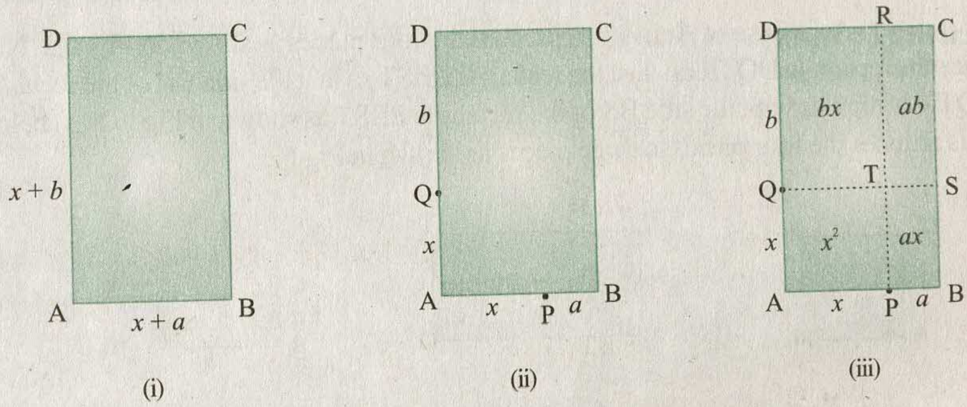


Fig. 6.1

Through P, draw a line segment PR parallel to AD meeting DC in R [Fig. 6.1(iii)]. Through Q, draw a line segment QS parallel to AB meeting BC in S [Fig. 6.1(iii)]. Let PR and QS meet in T. This divides the rectangle into the following four parts :

I. Square APTQ with side x, and area x^2

II. Rectangle PBST with sides a and x, and area ax

III. Rectangle QTRD with sides b and x, and area bx

IV. Rectangle TSCR with sides a and b, and area ab

Separate these four pieces by cutting along the line segments PR, QT and TS (Fig.6.2).

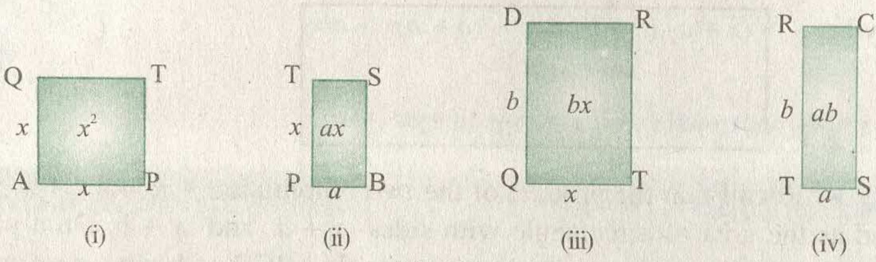


Fig. 6.2

Since the area of the whole piece must be the same as the total area of the separated pieces, we must have

$$(x + a)(x + b) = x^2 + ax + bx + ab$$

This verifies Identity I.

Activity 2 : As in case of Activity 1, prepare the four pieces as shown in Fig. 6.2. Now place the rectangle DQTR against the rectangle PBST so that the side DR of the rectangle DQTR conicides with the side BS of the rectangle PBST as shown in Fig. 6.3(ii) below. This reduces the four pieces to three pieces as shown in Fig.6.3.

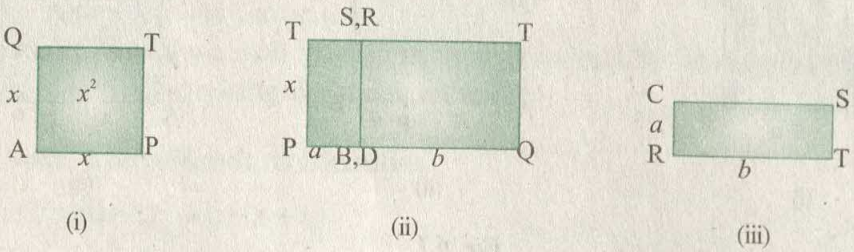


Fig. 6.3

It is clear that the areas of the respective pieces are x^2, $(a + b)\,x$ and ab. Since the area of the whole piece should be the same as the total area of the separated and the reassembled pieces, we must have

$$(x + a)(x + b) = x^2 + (a + b)\,x + ab$$

In the two activities above, we started with a whole piece and cut it out to verify Identity I. We could have done it the other way round as shown in Activities 3 and 4 below.

Activity 3 : Choose three numbers x, a and b, say $x = 2$ cm, $a = 3$ cm and $b = 4$ cm. From a piece of cardboard (or an old greeting card), cut out four pieces of dimensions as shown in Fig. 6.4.

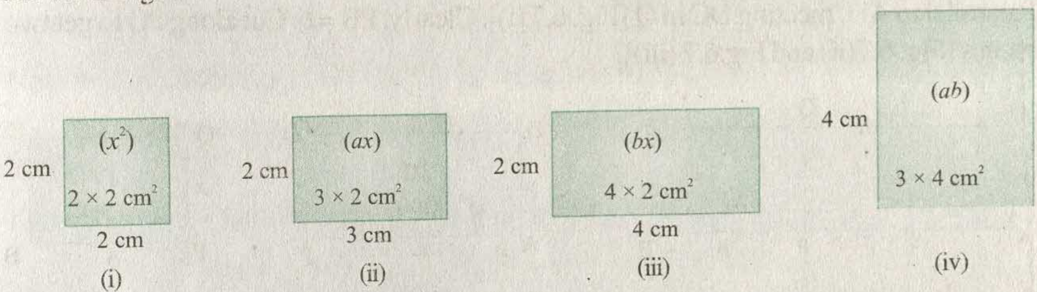


Fig. 6.4

The respective areas of these pieces are shown inside the pieces. The total area of the four pieces is $(2^2 + 3 \times 2 + 4 \times 2 + 3 \times 4)$ cm^2 i.e., $(x^2 + ax + bx + ab)$ cm^2. Now assemble these pieces so that the same form a rectangle of sides $(2 + 3)$ cm and $(2 + 4)$ cm i.e., $(x + a)$ cm and $(x + b)$ cm. One way to do this is shown in Fig. 6.5.

Since the total area of the pieces is equal to the area of the assembled pieces, therefore, we must have

$(x + a)(x + b) = x^2 + ax + bx + ab$, where $x = 2$ cm, $a = 3$ cm, $b = 4$ cm.

This verifies Identity I as before.

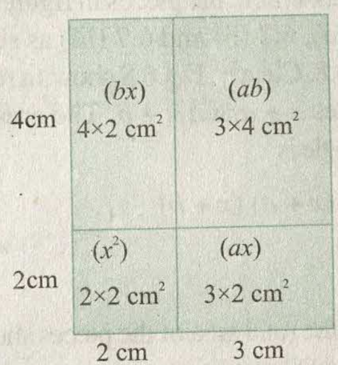


Fig. 6.5

Activity 4 : Start with three pieces as shown below in Fig. 6.6.

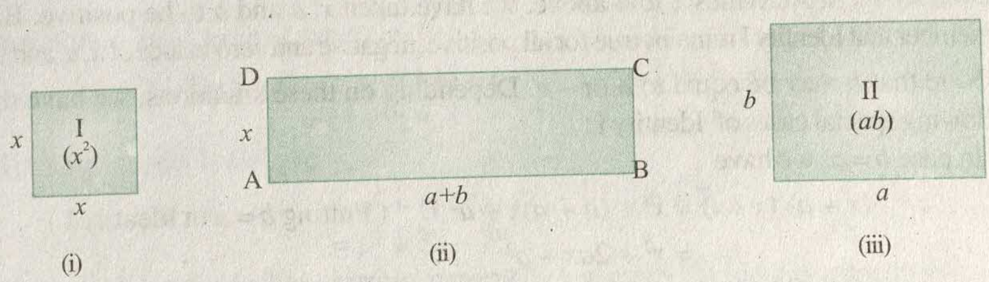


Fig. 6.6

Observe that the total area of these three pieces is

$$x^2 + (a + b)x + ab \qquad (1)$$

Mark a point P on AB in Fig. 6.6 (ii) so that AP = a. Draw a line segment through P parallel to AD, meeting DC in Q [Fig. 6.7 (i)]. Clearly, PB = b. Cut along PQ to get two pieces [Fig. 6.7(ii) and Fig.6.7 (iii)].

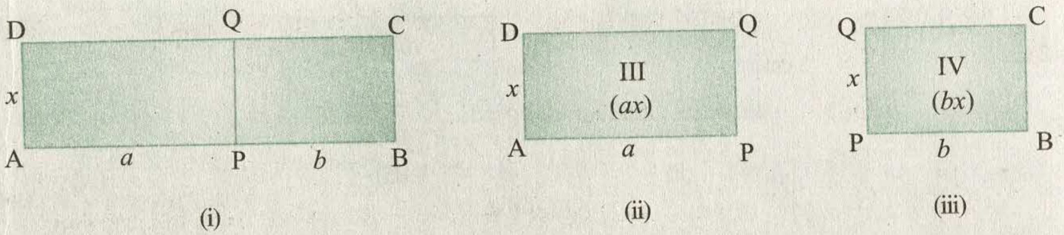


Fig. 6.7

Now assemble the pieces in figures 6.6 (i), 6.6 (iii), 6.7 (ii) and 6.7 (iii) as shown in Fig. 6.8. Clearly, Fig.6.8 shows a rectangle of sides $x + a$ and $x + b$. The area of this rectangle is

$$(x + a)(x + b) \qquad (2)$$

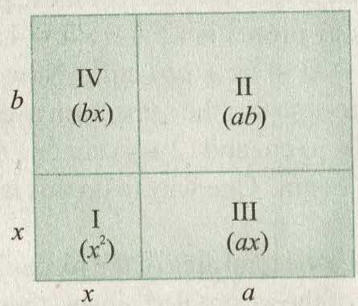


Fig. 6.8

Since the total area of the pieces should be the same whether assembled or separate, from (2) and (1), we have

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

This verifies Identity I as before.

Remarks 1 : In Activities 1 to 4 above, we have taken x, a and b to be positive. But remember that Identity I remains true for all positive, negative and zero values of x, a and b.

2. Note that b may be equal to a or $-a$. Depending on these situations, we have the following special cases of Identity I :

(i) In case $b = a$, we have

$$(x + a)(x + a) = x^2 + (a + a)x + a^2 \qquad \text{(Putting } b = a \text{ in Identity I)}$$

$$= x^2 + 2ax + a^2$$

Note that this is the same as Identity A in Section 6.1.

(ii) In case $b = -a$, we have,

$$(x + a)\{x + (-a)\} = x^2 + \{a + (-a)\}x + a \times (-a),$$

(Putting $b = -a$ in Identity I)

$$= x^2 - a^2$$

Note that this is the same as Identity C in Section 6.1.

We shall now show the use of Identity I in simplifying algebraic expressions and evaluating products.

Example 1 : Find the following products using Identity I :

(i) $(y + 2)(y + 6)$ (ii) $(x + 5)(x - 3)$

Solution : (i) Let us compare $(y + 2)(y + 6)$ with $(x + a)(x + b)$. We observe that

$$x = y,\ a = 2,\ b = 6$$

Therefore, using Identity I, we have

$$(y + 2)(y + 6) = y^2 + (2 + 6)y + 2 \times 6$$
$$= y^2 + 8y + 12$$

(ii) Let us compare $(x + 5)(x - 3)$ with $(x + a)(x + b)$. We observe that

$$a = 5,\ b = -3$$

Therefore, using Identity I, we have

$$(x + 5)(x - 3) = x^2 + \{5 + (-3)\}x + 5 \times (-3)$$
$$= x^2 + 2x - 15$$

Example 2 : Find the following products :

(i) $(z - 1)(z - 8)$ (ii) $(p - 4)(p + 7)$

Solution : (i) Let us compare $(z - 1)(z - 8)$ with $(x + a)(x + b)$. We observe that

$$x = z,\ a = -1,\ b = -8$$

Therefore, using Identity I, we have

$$(z - 1)(z - 8) = z^2 + \{(-1) + (-8)\}z + (-1) \times (-8)$$
$$= z^2 - 9z + 8$$

(ii) Using Identity I, we have

$$(p - 4)(p + 7) = p^2 + \{(-4) + 7\}p + (-4) \times 7$$
$$= p^2 + 3p - 28$$

Remark : You may do the comparison mentally and apply the Identity directly as in part (ii) above.

Example 3 : Evaluate 104×106 without directly multiplying the two given numbers.

Solution : Writing 104 as $100 + 4$ and 106 as $100 + 6$, we get

$$\begin{aligned}
104 \times 106 &= (100 + 4) \times (100 + 6) \\
&= 100^2 + (4 + 6)\,100 + 4 \times 6 \qquad \text{(Using Identity I)} \\
&= 10000 + 10 \times 100 + 24 \\
&= 10000 + 1000 + 24 \\
&= 11024
\end{aligned}$$

Example 4 : Find the value of 83×79 by using a suitable Identity.

Solution :

$$\begin{aligned}
83 \times 79 &= (80 + 3) \times (80 - 1) \\
&= 80^2 + (3 - 1)\,80 + 3 \times (-1) \qquad \text{(Using Identity I)} \\
&= 6400 + 160 - 3 \\
&= 6557
\end{aligned}$$

EXERCISE 6.1

Use a suitable identity to find the following products :

1. $(x + 4)(x + 5)$ **2.** $(x + 6)(x + 9)$

3. $(x + 8)(x + 7)$ **4.** $(x + 4)(x + 9)$

5. $(x + 2)(x + 6)$ **6.** $(x + 1)(x - 1)$

7. $(p + 6)(p - 4)$ **8.** $(y + 8)(y - 3)$

9. $(x - 4)(x - 1)$ **10.** $(z - 14)(z - 1)$

11. $(y - 4)(y - 11)$ **12.** $(x - 4)(x + 21)$

13. $(x - 7)(x + 12)$ **14.** $(y - 4)(y + 20)$

Evaluate the following products :

15. (i) $\left(x + \frac{1}{5}\right)(x + 5)$ (ii) $(y + 6)\left(y + \frac{5}{12}\right)$

16. (i) $\left(z + \frac{3}{4}\right)\left(z + \frac{4}{3}\right)$ (ii) $(x^2 + 4)(x^2 + 9)$

17. (i) $(y^2 + 12)(y^2 + 6)$ (ii) $(q^2 + 4)(q^2 - 1)$

18. (i) $(p^2 + 16)\left(p^2 - \frac{1}{4}\right)$ (ii) $\left(y^2 + \frac{5}{7}\right)\left(y^2 - \frac{14}{5}\right)$

19. (i) $(z^3 + 14)(z^3 + 1)$ (ii) $(z^3 + 1)(z^3 - 8)$

20. (i) $(y^3 + 2)\left(y^3 - \frac{3}{8}\right)$ (ii) $\left(x^3 - \frac{3}{8}\right)\left(x^3 + \frac{16}{17}\right)$

Evaluate the following products without directly multiplying the given numbers :

21. (i) 103×106 (ii) 204×207

22. (i) 95×96 (ii) 86×82

23. (i) 98×103 (ii) 95×101

24. (i) 194×189 (ii) 205×192

25. (i) 198×209 (ii) 204×197

II. Expansion of $(a + b + c)^2$

You have already learnt how to expand the square of a binomial $(a + b)$. We may extend this notion to expand the square of a trinomial $(a + b + c)$ as follows :

Let $b + c = x$. Then

$$
\begin{aligned}
(a + b + c)^2 &= (a + x)^2 \\
&= a^2 + 2ax + x^2 && \text{(Using Identity A)} \\
&= a^2 + 2a(b + c) + (b + c)^2 && \text{(Since } x = b + c\text{)} \\
&= a^2 + 2ab + 2ac + (b^2 + 2bc + c^2) && \text{(Using Identity A)} \\
&= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca && \text{(Rearranging terms)}
\end{aligned}
$$

Thus, we have the following identity :

Identity II : $\boxed{(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca}$

Remarks 1 : Note that the expansion of the square of the expression $a + b + c$ consists of three square terms and three product terms.

2. a, b and c may take any positive, negative values or can be zero.

Activity 5 : We shall now verify the identity above geometrically and experimentally. On a piece of cardboard (or an old greeting card), draw a square ABCD with sides $a + b + c$ [Fig. 6.9 (i)] taking suitable values of a, b and c.

Clearly, the area of this square is $(a + b + c)^2$. Mark two points P and Q on AB so that $AP = a$ and $PQ = b$ [Fig. 6.9 (ii)]. Clearly, this means $QB = c$.

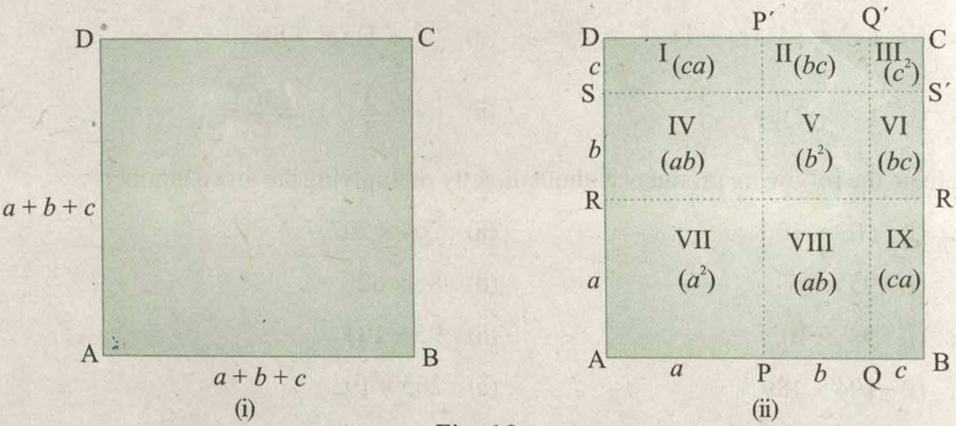


Fig. 6.9

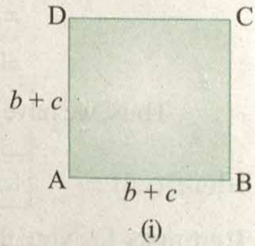
Mark two points R and S on AD so that AR = a and RS = b [Fig. 6.9(ii)]. Clearly, this means SD = c. Through P and Q, draw line segments PP′ and QQ′ parallel to AD meeting DC in P′ and Q′, respectively. Through R and S, draw line segments RR′ and SS′ parallel to AB meeting BC in R′ and S′, respectively. This divides the square into nine parts marked I, II, . . . , IX. The areas of these parts are ca, bc, c^2, ab, b^2, bc, a^2, ab and ca, respectively. The total area of these parts is

$$ca + bc + c^2 + ab + b^2 + bc + a^2 + ab + ca, \text{ i.e., } a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

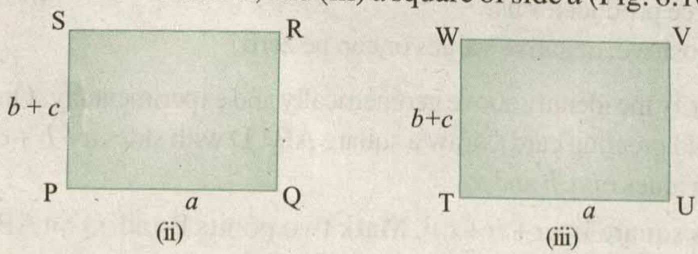
Since the area of the whole is the sum total of the areas of the parts, we must have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

This verifies the Identity geometrically.



We shall now verify this Identity experimentally. Take different values for a, b and c lying in the range of 1cm to 8 cm so that the pieces may be handled conveniently. This restriction is, however, optional. From a piece of cardboard (or an old greeting card), cut out (i) a square of side $b + c$, (ii) two rectangles of sides $b + c$ and a, and (iii) a square of side a (Fig. 6.10).



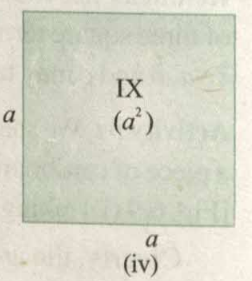


Fig. 6.10

Consider Fig.6.10(i). Mark a point on AB at a distance b from A [Fig. 6.11(i)]. Through the marked point, draw a line parallel to AD. Also mark a point on AD at a distance b from A. Through the marked point, draw a line parallel to AB. This divides the square ABCD into four parts shown as I, II, III and IV [Fig. 6.11 (i)]. The dimensions of the parts are shown in the figure. Cut out along the drawn lines to separate the parts.

Consider Fig. 6.10 (ii). Mark a point on SP at a distance b from S [Fig. 6.11(ii)]. Through the marked point, draw a line parallel to SR. This divides the rectangle PQRS into two parts shown as V and VI [Fig. 6.11(ii)]. The dimensions of the parts are shown in the figure. Cut out along the drawn line to separate the two parts.

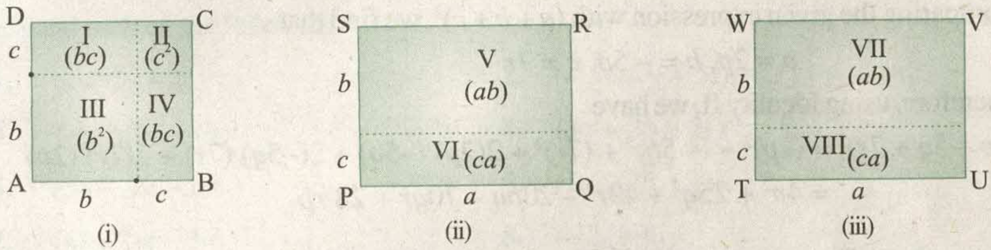


Fig. 6.11

Consider Fig. 6.10 (iii). Mark a point on WT at a distance b from W [Fig. 6.11 (iii)]. Through the marked point, draw a line parallel to WV. This divides the rectangle TUVW into two parts shown as VII and VIII [Fig. 6.11 (iii)]. The dimensions of the parts are shown in the figure. Cut out along the drawn line to separate the two parts.

Label the square with side a [Fig. 6.10 (iv)] as IX. Now assemble the parts I to IX as shown in Fig. 6.12. It is clear that what we get is a square of side $a+b+c$. The area of this square is obviously $(a + b + c)^2$.

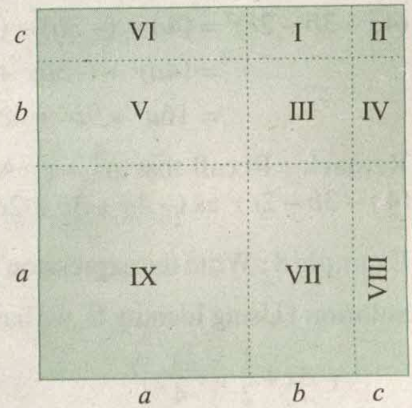


Fig. 6.12

Since the area of the whole should be the same as the total area of the parts, we have

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

We shall now show the use of Identity II in writing the square of a trinomial in expanded form.

Example 5 : Write $(7x + 4y + 3z)^2$ in expanded form.

Solution : Comparing the given expression with $(a + b + c)^2$, we find that

$$a = 7x,\ b = 4y,\ c = 3z$$

Therefore, using Identity II, we have

$$(7x + 4y + 3z)^2 = (7x)^2 + (4y)^2 + (3z)^2 + 2(7x)(4y) + 2(4y)(3z) + 2(3z)(7x)$$
$$= 49x^2 + 16y^2 + 9z^2 + 56xy + 24yz + 42zx$$

Example 6 : Expand $(2p - 5q + 7r)^2$.

Solution : We can write the given expression $(2p - 5q + 7r)^2$ as $[2p + (-5q) + 7r]^2$. Comparing the given expression with $(a + b + c)^2$, we find that

$$a = 2p,\ b = -5q,\ c = 7r$$

Therefore, using Identity II, we have

$$(2p - 5q + 7r)^2 = (2p)^2 + (-5q)^2 + (7r)^2 + 2(2p)(-5q) + 2(-5q)(7r) + 2(7r)(2p)$$
$$= 4p^2 + 25q^2 + 49r^2 - 20pq - 70qr + 28\,rp$$

Example 7 : Expand $(4a - 3b - 2c)^2$.

Solution : Using Identity II, we have

$$(4a - 3b - 2c)^2 = [4a + (-3b) + (-2c)]^2$$
$$= (4a)^2 + (-3b)^2 + (-2c)^2 + 2(4a)(-3b) + 2(-3b)(-2c) + 2(-2c)(4a)$$
$$= 16a^2 + 9b^2 + 4c^2 - 24ab + 12bc - 16ac$$

Remark : Recall that $A^2 = (-A)^2$. Thus, we could have evaluated the expression $(4a - 3b - 2c)^2$ as $(-4a + 3b + 2c)^2$ also.

Example 8 : Write the expression $\left(-5x + \frac{1}{2}y + \frac{3}{4}z\right)^2$ in expanded form.

Solution : Using Identity II, we have

$$(-5x + \frac{1}{2}y + \frac{3}{4}z)^2$$
$$= (-5x)^2 + \left(\frac{1}{2}y\right)^2 + \left(\frac{3}{4}z\right)^2 + 2(-5x)\left(\frac{1}{2}y\right) + 2\left(\frac{1}{2}y\right)\left(\frac{3}{4}z\right) + 2\left(\frac{3}{4}z\right)(-5x)$$
$$= 25x^2 + \frac{1}{4}y^2 + \frac{9}{16}z^2 - 5xy + \frac{3}{4}yz - \frac{15}{2}xz$$

Example 9 : Simplify $(x + 2y - 3z)^2 + (x - 2y + 3z)^2$.

Solution : Using Identity II,

$$(x + 2y - 3z)^2 = x^2 + 4y^2 + 9z^2 + 4xy - 12yz - 6xz \quad (1)$$

$$(x - 2y + 3z)^2 = x^2 + 4y^2 + 9z^2 - 4xy - 12yz + 6xz \quad (2)$$

Adding the corresponding sides of (1) and (2), we get

$$(x + 2y - 3z)^2 + (x - 2y + 3z)^2 = 2(x^2 + 4y^2 + 9z^2 - 12yz)$$

EXERCISE 6.2

Expand each of the following:

1. $(x + 2y + 4z)^2$

2. $(-3x + y + 5z)^2$

3. $(-x - 2y + 6z)^2$

4. $(3a + 2b - 3c)^2$

5. $(3a - 7b - c)^2$

6. $(5a - 7b + c)^2$

7. $(4l + 2m - 3n)^2$

8. $(-2l + m - 8n)^2$

9. $(l + 2m - 7n)^2$

10. $(p + 9q + 2)^2$

11. $\left(6x + \frac{1}{2}y + 4z\right)^2$

12. $\left(9x - y + \frac{1}{3}z\right)^2$

13. $\left(\frac{1}{4}a - \frac{1}{2}b + 16\right)^2$

14. $\left(-a - \frac{1}{2}b - 6\right)^2$

Fill in the blanks so as to make each of the following statements true :

15. $(3x - 4y + 2z)^2 = ...x^2 + ...y^2 + ...z^2 - ...xy - ...yz + ...zx$

16. $(-2x - 3y + 5z)^2 = ...x^2 + ...y^2 + ...z^2 + ...xy - ...yz - ...zx$

17. $(a - b + c)^2 = a^2 ... b^2 ... c^2 ... 2ab ... 2bc ... 2ca$

18. $(a - 2b + 7c)^2 = a^2 ... b^2 ... c^2 ... 4ab ... 28bc ... 14ca$

Simplify :

19. $(p + q + r)^2 + (p - q - r)^2$

20. $(p - q + r)^2 + (p - q - r)^2$

21. $(p + q + r)^2 - (p - q - r)^2$

22. $(p - q + r)^2 - (p - q - r)^2$

23. $(2x + y + z)^2 - (2x - y - z)^2$

24. $(2x - y + z)^2 - (2x + y - z)^2$

III. Expansion of $(a+b)^3$

You have already learnt how to expand the square of a binomial $(a+b)$. We may extend this notion to expand the cube of a binomial $(a+b)$ as follows :

By Identity A, we have

$$(a+b)^2 = a^2 + 2ab + b^2$$

Multiplying both sides by $(a+b)$, we get

$$(a+b)(a+b)^2 = (a+b)(a^2 + 2ab + b^2)$$

or $\quad (a+b)^3 = a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$

$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$ (Combining like terms and arranging in descending powers of a)

Thus, we have the following identity :

Identity III : $\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$

We may rearrange the terms on the RHS of the Identity III above in a form that is easier to remember.

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$= a^3 + b^3 + 3a^2b + 3ab^2$ (Rearranging terms)

$= a^3 + b^3 + 3ab(a+b)$ (Taking $3ab$ as a common factor from the last two terms)

This gives us the following alternate form of Identity III.

Identity III′: $\boxed{(a+b)^3 = a^3 + b^3 + 3ab(a+b)}$

Geometric Verification of the Identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

In the previous class, you learnt how to make paper cubes and cuboids. To verify the above identity, you would need to make

- A cube of side a (to be denoted as $a \times a \times a$ cube)
- A cube of side b (to be denoted as $b \times b \times b$ cube)
- Three cuboids of sides a, a, b (to be denoted as $a \times a \times b$ cuboids)
- Three cuboids of sides a, b, b (to be denoted as $a \times b \times b$ cuboids)

Of these eight pieces, first take

- the $a \times a \times a$ cube
- two of the $a \times a \times b$ cuboids, and
- one $a \times b \times b$ cuboid.

Place these four pieces so that the bases are as shown in Fig. 6.13(i). Then the height of each piece will be a [Fig. 6.13 (ii)]. Clearly, the bases form a square of side $a + b$. The four blocks form the bottom layer of a solid that we are going to make. Notice that this layer is a cuboid of sides $a + b$, $a + b$ and a.

Take the remaining four pieces and place them so that their bases are also as shown in Fig. 6.13(i). The height of each piece will then be b [Fig. 6.13 (iii)]. Stack these pieces on top of the bottom layer [Fig. 6.13 (ii)] so that there are no gaps. This forms the top layer. Note that this layer is a cuboid with sides $a + b$, $a + b$ and b. Its base conicides with the top of the bottom layer. Since the bottom layer has height a and the top layer has height b, therefore, the height of the solid so formed is $a + b$ [Fig. 6.13 (iv)]. Thus, the eight pieces with which we started have been assembled into a cube of side $a + b$.

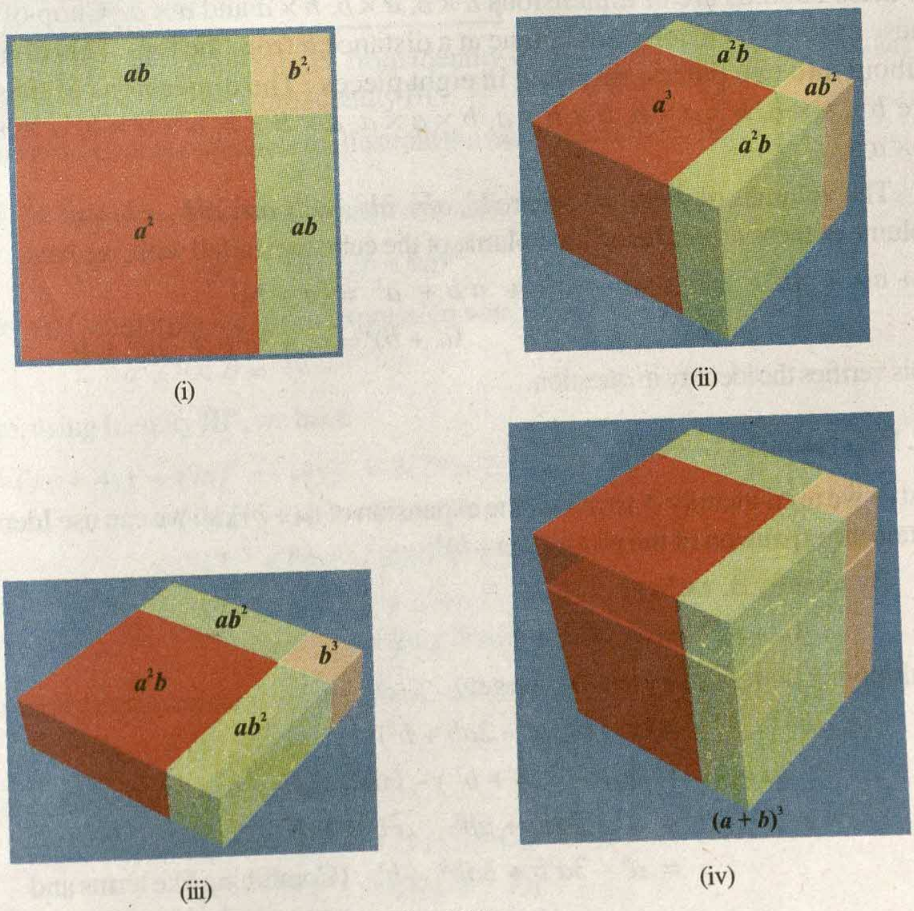


Fig. 6.13

Since there are no gaps and obviously no overlaps, the volume of the solid formed is the same as the total volume of the pieces. Hence, we have

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

This verifies the identity geometrically.

We can reverse the process and start with a cube of side $a + b$. Take a solid chunk of some soft but firm vegetable like a gourd or potato. Chop off on all sides to turn it into a cube. Take a small piece of a matchstick of length b say. At one corner, measure off this distance on the top face of the vegetable chunk in both directions. Slice off the vegetable chunk by means of vertical planes at the distance b marked off at the corner. This divides the cube into four cuboids of height $a + b$. The top faces of these cuboids are of dimensions $b \times b$, $a \times b$, $b \times a$ and $a \times a$. Chop off each of these cuboids by a horizontal plane at a distance b from the top. This divides each cuboid into two pieces, resulting in eight pieces. The dimensions of these pieces are $b \times b \times b$, $b \times a \times b$, $b \times b \times a$, $b \times a \times a$, $a \times b \times b$, $a \times a \times b$, $a \times b \times a$ and $a \times a \times a$.

The volumes of these pieces are b^3, ab^2, ab^2, a^2b, ab^2, a^2b, a^2b, and a^3. The total volume of these pieces being the volume of the cube we started with, we have

$$b^3 + ab^2 + ab^2 + a^2b + ab^2 + a^2b + a^2b + a^3 = (a + b)^3$$

or $$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

This verifies the identity in question.

IV. Expansion of $(a - b)^3$

Just as we used Identity A to obtain the expansion of $(a + b)^3$, so we can use Identity B to obtain an expansion of the product $(a - b)^3$.

By Identity B, we have

$$(a - b)^2 = a^2 - 2ab + b^2$$

Multiplying both sides by $(a - b)$, we get

$$(a - b)(a - b)^2 = (a - b)(a^2 - 2ab + b^2)$$

or

$$\begin{aligned}(a - b)^3 &= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2)\\ &= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3\\ &= a^3 - 3a^2b + 3ab^2 - b^3\end{aligned}$$

(Combining like terms and arranging in descending powers of a)

Thus, we have the following identity:

Identity IV : $\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$

We may rearrange the terms on the RHS of Identity IV above in a form that is easier to remember. We have

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$= a^3 - b^3 - 3a^2b + 3ab^2 \quad \text{(Rearranging terms)}$$

$$= a^3 - b^3 - 3ab\,(a-b) \quad \text{(Taking } -3ab \text{ as a common factor from the last two terms)}$$

This gives us the following alternate form of Identity IV :

Identity IV′ : $\boxed{(a-b)^3 = a^3 - b^3 - 3ab\,(a-b)}$

Remark : We may obtain Identity IV from Identity III by replacing b by $-b$. Similarly, Identity IV′ may be obtained from Identity III′.

We now take some examples to illustrate the use of these identities.

Example 10 : Write the following cubes in expanded form :

(i) $(7x + 4y)^3$ (ii) $(2p - 9q)^3$

Solution : (i) Comparing the given expression with $(a + b)^3$, we find that

$$a = 7x,\ b = 4y$$

Therefore, using Identity III′, we have

$$(7x + 4y)^3 = (7x)^3 + (4y)^3 + 3(7x)(4y)(7x + 4y)$$

$$= 343x^3 + 64y^3 + 84xy(7x + 4y)$$

$$= 343x^3 + 64y^3 + 588x^2y + 336xy^2$$

$$= 343x^3 + 588x^2y + 336xy^2 + 64y^3$$

(Rearranging in descending powers of x)

(ii) Comparing the given expression with $(a - b)^3$, we find that

$$a = 2p,\ b = 9q$$

Therefore, using Identity IV′, we have

$$(2p - 9q)^3 = (2p)^3 - (9q)^3 - 3(2p)(9q)(2p - 9q)$$

$$= 8p^3 - 729q^3 - 54pq\,(2p - 9q)$$

$$= 8p^3 - 729q^3 - 108p^2q + 486pq^2$$

$$= 8p^3 - 108p^2 q + 486pq^2 - 729q^3$$

(Rearranging in descending powers of p)

Example 11 : Expand (i) $(-3x + 5y)^3$ and (ii) $(-2z - y)^3$.

Solution : (i) Using Identity III′, we have

$$\begin{aligned}(-3x + 5y)^3 &= (-3x)^3 + (5y)^3 + 3(-3x)(5y)(-3x + 5y)\\ &= -27x^3 + 125y^3 - 45xy(-3x + 5y)\\ &= -27x^3 + 125y^3 + 135x^2y - 225xy^2\\ &= -27x^3 + 135x^2y - 225xy^2 + 125y^3\end{aligned}$$

(Rearranging in descending powers of x)

(ii) Using Identity III′,

$$\begin{aligned}(-2z - y)^3 &= \{(-2z) + (-y)\}^3\\ &= (-2z)^3 + (-y)^3 + 3(-2z)(-y)(-2z - y)\\ &= -8z^3 - y^3 + 6yz(-2z - y)\\ &= -8z^3 - y^3 - 12yz^2 - 6y^2z\\ &= -y^3 - 6y^2z - 12yz^2 - 8z^3\end{aligned}$$

(Rearranging in descending powers of y)

$$= -(y^3 + 6y^2z + 12yz^2 + 8z^3)$$

Remark : Notice that $(-A)^3 = -A^3$. Thus, we could write $(-2z - y)^3$ as $-(2z + y)^3$ also.

Example 12: Simplify $(x + 4y)^3 - (x - 4y)^3$.

Solution : Using Identity III′,

$$\begin{aligned}(x + 4y)^3 &= (x)^3 + (4y)^3 + 3(x)(4y)(x + 4y)\\ &= x^3 + 64y^3 + 12xy(x + 4y)\\ &= x^3 + 64y^3 + 12x^2y + 48xy^2\\ &= x^3 + 12x^2y + 48xy^2 + 64y^3\end{aligned}$$

(Rearranging in descending powers of x)

Using Identity IV′,

$$\begin{aligned}(x - 4y)^3 &= (x)^3 - (4y)^3 - 3(x)(4y)(x - 4y)\\ &= x^3 - 64y^3 - 12xy(x - 4y)\\ &= x^3 - 64y^3 - 12x^2y + 48xy^2\end{aligned}$$

$= x^3 - 12x^2y + 48xy^2 - 64y^3$

(Rearranging in descending powers of x)

Using the values of $(x + 4y)^3$ and $(x - 4y)^3$ found above, we have

$(x + 4y)^3 - (x - 4y)^3 = (x^3 + 12x^2y + 48xy^2 + 64y^3) - (x^3 - 12x^2y + 48xy^2 - 64y^3)$

$= 24x^2y + 128y^3$

Example 13 : Find the value of $x^3 + 8y^3$, if $x + 2y = 8$ and $xy = 6$.

Solution : We are given that

$x + 2y = 8$ and $xy = 6$.

Now $(x + 2y)^3 = x^3 + (2y)^3 + 3(x)(2y)(x + 2y)$ (Using Identity III′)

$= x^3 + 8y^3 + 6xy(x + 2y)$

$\therefore$ $x^3 + 8y^3 = (x + 2y)^3 - 6xy(x + 2y)$

$= (8)^3 - 6(6)(8)$ (Substituting $x + 2y = 8$ and $xy = 6$)

$= 512 - 288$

$= 224$

Example 14 : Evaluate 1001^3 by making use of a suitable Identity.

Solution : Using Identity III′, we have

$1001^3 = (1000 + 1)^3$

$= 1000^3 + 1^3 + 3(1000)(1)(1000 + 1)$

$= 1000000000 + 1 + 3000(1000 + 1)$

$= 1000000000 + 1 + 3000000 + 3000$

$= 1003003001$

Example 15 : Evaluate 998^3 by using some suitable Identity.

Solution : Using Identity IV′, we have

$998^3 = (1000 - 2)^3$

$= (1000)^3 - (2)^3 - 3(1000)(2)(1000 - 2)$

$= (1000)^3 - (2)^3 - 6(1000)(998)$

$= 1000000000 - 8 - 5988000$

$= (1000000000 - 5988000) - 8$

$= 994012000 - 8$

$= 994011992$

EXERCISE 6.3

Expand each of the following :

1. $(x + 2y)^3$ **2.** $(2x - 3y)^3$ **3.** $(ax + by)^3$

4. $(x^2 + 2y)^3$ **5.** $(2x - y^2)^3$ **6.** $(-x + 4y)^3$

7. $(a + 5y)^3$ **8.** $\left(\frac{1}{3}x+\frac{5}{3}y\right)^3$ **9.** $\left(\frac{1}{3}x-\frac{2}{3}y\right)^3$

10. Find the value of $8x^3 + 27y^3$, if

(i) $2x + 3y = 8$ and $xy = 2$ (ii) $2x + 3y = 18$ and $xy = 12$

(iii) $2x + 3y = 19$ and $xy = 3$ (iv) $2x+3y = \frac{21}{2}$ and $xy = \frac{5}{6}$

11. Find the value of $p^3 - 8y^3$, if

(i) $p - 2y = 2$ and $py = 8$ (ii) $p - 2y = 1$ and $py = 10$

(iii) $p - 2y = 13$ and $py = -21$ (iv) $p - 2y = -11$ and $py = -5$

12. Find the value of $125p^3 - 8q^3$, if

(i) $5p - 2q = 1$ and $pq = 2$ (ii) $5p - 2q = 6$ and $pq = 4$

(iii) $5p - 2q = 7$ and $pq = 12$ (iv) $5p - 2q = 13$ and $pq = 30$

Simplify :

13. $(a + 2b)^3 + (a - 2b)^3$ **14.** $(a - 3b)^3 + (a + 3b)^3$

15. $(2a + 5b)^3 - (2a - 5b)^3$ **16.** $(7 - 2b)^3 - (7 + 2b)^3$

17. $\left(\frac{1}{3}a+\frac{2}{3}b\right)^3+\left(\frac{1}{3}a-\frac{2}{3}b\right)^3$ **18.** $\left(\frac{1}{3}a+\frac{2}{3}b\right)^3-\left(\frac{1}{3}a-\frac{2}{3}b\right)^3$

Evaluate by using a suitable Identity :

19. (i) $(104)^3$ (ii) $(1004)^3$ (iii) $(503)^3$

20. (i) $(99)^3$ (ii) $(996)^3$ (iii) $(999)^3$

21. (i) $(599)^3$ (ii) $(9.8)^3$ (iii) $(8.01)^3$

6.3 Factorisation of Algebraic Expressions

Recall that often it is possible to write a given algebraic expression as the product of two or more algebraic expressions (and numbers). When an algebraic epxression is expressed as the product of some numbers and algebraic expressions, then each of these numbers and algebraic expressions is called a *factor* of the given expression. The process of writing the

given expression as the product of numbers and algebraic expressions is called *factorisation*. For example, since $10\,pq = 5 \times 2 \times p \times q$, therefore, 5, 2, p and q are factors of $10\,pq$.

In Class VII, you learnt three basic methods of factorisation :

- Factorisation by taking out a common factor
- Factorisation by regrouping terms
- Factorisation by using Identities

You already know how to use Identities A, B and C (mentioned at the beginning of this Chapter) to factorise algebraic expressions. We shall now use Identities I to IV learnt earlier in this Chapter to factorise algebraic expressions.

Example 16 : Factorise $x^2 + 5x + 6$.

Solution : We know from Identity I that

$$(x + a)\,(x + b) = x^2 + (a + b)\,x + ab$$

This means that the given expression may be factorised as $(x + a)\,(x + b)$, if we are able to find two numbers a and b (positive or negative) such that the sum $a + b$ is the same as the coefficient of x and the product ab is equal to the constant term in the given expression. Therefore, let us try to find two numbers a and b such that

$a + b = 5$ (the coefficient of x)

and $ab = 6$ (the constant term)

Now factors of 6 are ± 1, ± 2, ± 3 and ± 6. A little *trial-and-error* with these numbers tells us that we may take a and b as 2 and 3. The sum of 2 and 3 is 5, and the product of 2 and 3 is 6. Hence,

$$x^2 + 5x + 6 = x^2 + (3 + 2)\,x + 3 \times 2 = (x + 3)\,(x + 2) \qquad \text{(Using Identity I)}$$

Remark : You could use simple logic to reduce the trial-and-error work for finding the values of a and b. Let us denote the sum $a + b$ by S and the product ab by P. Now notice that in the above example, S(5) is positive and so is P(6). We express this fact as

S : +, P : +

Since P is positive, therefore, either both a and b are positive or both a and b are negative. Since S is positive, both a and b are positive. So we do not consider the negative factors. It is easy to see that a and b have the values 2 and 3 in some order.

Example 17 : Factorise $x^2 + 3x - 10$.

Solution : Here, we have to find two numbers a and b such that

$a + b = 3$ (the coefficient of x) and $ab = -10$ (the constant term).

Now factors of -10 are ± 1, ± 2, ± 5 and ± 10. A little experimentation with these

numbers tells us that we may take a and b as 5 and -2. The sum of 5 and -2 is 3, and the product of 5 and -2 is -10. Hence,

$$x^2 + 3x - 10 = x^2 + \{5 + (-2)\}x + 5(-2)$$
$$= (x + 5)(x - 2) \qquad \text{(Using Identity I)}$$

Note : Here, S : +, P : –. Since P is negative, one of a and b is positive and the other is negative. With this in mind, we find that S being positive, the greater of a and b is positive. It is now easy to pick the values of a and b as 5 and -2.

Example 18 : Factorise $x^2 - 7x + 12$.

Solution : Here, S: –, P: +. This means that both a and b are negative. Since,

$$a + b = -7,\ ab = 12,$$

and negative factors of 12 are $-1, -2, -3, -4, -6$ and -12, we find that $a = -4$ and $b = -3$ (or $a = -3$ and $b = -4$). Hence,

$$x^2 - 7x + 12 = x^2 + \{(-4) + (-3)\}x + (-4) \times (-3)$$
$$= (x - 4)(x - 3) \qquad \text{(Using Identity I)}$$

Example 19 : Factorise $x^2 - 3x - 10$.

Solution : Here, S : –, P : –. Therefore, one of a and b is positive and the other is negative. S being negative, the numerically greater term is negative. Factors of 10 are $\pm 1, \pm 2, \pm 5$ and ± 10. Hence, the values -5 and 2 of a and b will serve the purpose. Thus,

$$x^2 - 3x - 10 = x^2 + \{2 + (-5)\}x + 2(-5)$$
$$= (x + 2)(x - 5) \qquad \text{(Using Identity I)}$$

Remark : It is not necessary to use Identity I to write the final factors if you get confused for some reason. Once you have come to the stage of breaking up the coefficient of x and the constant term, you may use regrouping. This enables you to take out a common factor. Thus, for the above example,

$$x^2 - 3x - 10 = x^2 + \{2 + (-5)\}x + 2(-5)$$
$$= x^2 + 2x + (-5)x + 2(-5)$$
$$= (x^2 + 2x) + \{(-5)x + 2(-5)\} \qquad \text{(Regrouping terms)}$$
$$= x(x + 2) + (-5)(x + 2) \qquad \text{(Taking out } x \text{ common from the first two terms and } -5 \text{ common from the last two terms)}$$
$$= (x + 2)(x - 5)$$

You may use this method to check your answer, if you have factorised the given expression using Identity I.

Example 20 : Factorise the expression :

$$4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$$

Solution : We notice that the first three terms are the squares of $2x$, y and z respectively. This suggests that we use Identity II, that is,

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Further, the terms with xy and yz are negative in the given expression. This may happen in one of the following two ways :

1. The literal y common to these two terms has a negative coefficient, and x and z have positive coefficients. In this case, the sequence of signs of coefficients of x, y, z is $+, -, +$.
2. y has a positive coefficient but both x and z have negative coefficients. In this case, the sequence of signs of coefficients of x, y, z is $-, +, -$.

Since the squares of an expression and its negative are the same, therefore, it does not matter which of the two cases above is taken. Taking the first case, we may write the given expression as follows :

$$4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz = (2x)^2 + (-y^2) + z^2 + 2(2x)(-y) + 2(-y)z + 2(2x)z$$

Comparing this form of the given expression with the RHS of Identity II, we find that

$$a = 2x,\ b = -y,\ c = z$$

Hence, $4x^2 + y^2 + z^2 - 4xy - 2yz + 4xz$

$$= (2x)^2 + (-y^2) + z^2 + 2(2x)(-y) + 2(-y)z + 2(2x)z$$

$$= \{2x + (-y) + z\}^2 \qquad \text{(Using Identity II)}$$

$$= (2x - y + z)^2$$

$$= (2x - y + z)(2x - y + z)$$

Example 21 : Factorise the expression $8x^3 + 27y^3 + 36x^2y + 54xy^2$.

Solution : We notice that the first two terms are the cubes of $2x$ and $3y$ respectively. Also, the remaining two terms have a factor 3. This suggests that we use Identity III′, that is,

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$$

We may write the given expression as follows :

$$8x^3 + 27y^3 + 36x^2y + 54xy^2 = (2x)^3 + (3y)^3 + 3(2x)(3y)(2x + 3y)$$

Comparing this form of the given expression with the RHS of Identity III′, we find that

$$a = 2x,\ b = 3y$$

Hence,

$$\begin{aligned} 8x^3 + 27y^3 + 36x^2y + 54xy^2 &= (2x)^3 + (3y)^3 + 3(2x)(3y)(2x + 3y) \\ &= (2x + 3y)^3 \qquad \text{(Using Identity III}'\text{)} \\ &= (2x + 3y)(2x + 3y)(2x + 3y) \end{aligned}$$

Example 22 : Factorise the expression $8x^3 - \frac{y^3}{27} - 2xy\left(2x - \frac{y}{3}\right)$.

Solution : We notice that $8x^3$ and $\frac{y^3}{27}$ are the cubes of $2x$ and $\frac{y}{3}$, respectively. This suggests that we use Identity IV′, that is,

$$(a - b)^3 = a^3 - b^3 - 3ab\,(a - b)$$

We may write the given expression as follows :

$$8x^3 - \frac{y^3}{27} - 2xy\left(2x - \frac{y}{3}\right) = (2x)^3 - \left(\frac{y}{3}\right)^3 - 3(2x)\left(\frac{y}{3}\right)\left(2x - \frac{y}{3}\right)$$

Comparing this form of the given expression with the RHS of Identity IV′, we find that

$$a = 2x,\ b = \frac{y}{3}$$

Hence,

$$\begin{aligned} 8x^3 - \frac{y^3}{27} - 2xy\left(2x - \frac{y}{3}\right) &= (2x)^3 - \left(\frac{y}{3}\right)^3 - 3(2x)\left(\frac{y}{3}\right)\left(2x - \frac{y}{3}\right) \\ &= \left(2x - \frac{y}{3}\right)^3 \qquad \text{(Using Identity IV}'\text{)} \\ &= \left(2x - \frac{y}{3}\right)\left(2x - \frac{y}{3}\right)\left(2x - \frac{y}{3}\right) \end{aligned}$$

Remark : Not every polynomial of the form $x^2 + ax + b$ can be factorised as shown in the examples above. You will see this in later chapters.

EXERCISE 6.4

Factorise each of the following expressions:

1. $x^2 + 10x + 9$

2. $x^2 + 7x + 12$

3. $y^2 - 2y - 8$

4. $y^2 - 6y - 7$

5. $p^2 + 3p - 4$

6. $p^2 + 4p - 12$

7. $m^2 - 8m + 15$

8. $m^2 - 10m + 24$

Factorise each of the following expressions :

9. $9x^2 + y^2 + 25z^2 + 6xy + 10yz + 30xz$

10. $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16xz$

11. $m^2 + 4n^2 + 25z^2 - 4mn - 20nz + 10mz$

12. $49m^2 + 4n^2 + 9z^2 - 28mn + 12nz - 42mz$

13. $9x^2 + y^2 + 25 + 6xy + 10y + 30x$

Factorise each of the following expressions :

14. $p^2 + \frac{q^2}{4} + 1 + pq + q + 2p$

15. $\frac{p^2}{4} + \frac{q^2}{9} + 36 + \frac{pq}{3} + 4q + 6p$

16. $2x^2 + y^2 + 8z^2 - 2\sqrt{2}xy - 4\sqrt{2}yz + 8xz$ [*Hint*: $2 = (\sqrt{2})^2$]

17. $3x^2 + 3y^2 + z^2 + 6xy + 2\sqrt{3}yz + 2\sqrt{3}xz$

Factorise each of the following expressions :

18. $8x^3 + y^3 + 12x^2y + 6xy^2$

19. $8x^3 - y^3 - 12x^2y + 6xy^2$

20. $27q^3 - 125p^3 - 135q^2p + 225qp^2$

21. $64p^3 - 27q^3 - 144p^2q + 108pq^2$

22. $27 - 125p^3 - 135p + 225p^2$

23. $64p^3 - 27 - 144p^2 + 108p$

24. $8x^3 + 729 + 108x^2 + 486x$

25. $27x^3 - \frac{1}{216} - \frac{9}{2}x^2 + \frac{1}{4}x$

Things to Remember

Some standard identities :

1. $(a + b)^2 = a^2 + 2ab + b^2$

2. $(a - b)^2 = a^2 - 2ab + b^2$

3. $(a + b)(a - b) = a^2 - b^2$

4. $(x + a)(x + b) = x^2 + ax + bx + ab$

or

$(x + a)(x + b) = x^2 + (a + b)x + ab$

5. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

6. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

or

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

7. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

or

$(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

CHAPTER

7

POLYNOMIALS

7.1 Introduction

In earlier classes, you have studied algebraic expressions. As you know, an algebraic expression may contain several literals. The literals in the algebraic expressions that you studied had non-negative integral exponents only. The algebraic expressions with this property are called *polynomials*. Some polynomials are simpler than other polynomials in the sense that these contain only one literal, say x. Such polynomials are called *polynomials in one variable* x.

In this Chapter, we shall learn about polynomials in one variable. We shall learn how to divide a polynomial by a monomial or a binomial. We shall also discuss two useful and interesting relations among the dividend, divisor, quotient and remainder that are obtained in this process of division.

7.2 Polynomials in One Variable

As mentioned in Section 7.1,

> A polynomial is an algebraic expression in which the variables have non-negative integral exponents only.

For example, all of the following expressions are polynomials :

$$17 + 2x + x^2,\ 7x^3 + \sqrt{2}x^2y - 5xy^2 + 12y^3,\ x^4 + 3x - 9$$

The first and the last are polynomials in one variable x. The middle expression is a polynomial in two variables x and y. Henceforth, unless stated otherwise, the word *polynomial will mean a polynomial in one variable*.

Is any of the following expressions a polynomial?

$$6 + 2x^{-2} + x^2,\ x^2 + 3\sqrt{x} - 9,\ 2x^2 - x^{\frac{1}{3}} + 4$$

In case of the first expression, the variable x appears with a negative exponent. In the second and the third expressions, the exponents of the variables are not necessarily positive integers. Hence, none of these is a polynomial.

Polynomials having only one term are known as *monomials*. Polynomials having only two terms are known as *binomials*. Polynomials having only three terms are known as *trinomials*. The part *poly* in polynomial means *many*. Thus, the word meaning of polynomial signifies an algebraic expression with many terms. Monomials, binomials and trinomials are also polynomials.

A polynomial that contains only one variable, say x, is known as a polynomial in the variable x.

Thus, $5x^2 + 13x - 9$ is a polynomial in the variable x.

$y^3 + 7y - 19$ is a polynomial in the variable y.

It is customary to write the terms of a polynomial in decreasing order of the exponents of the variables. This is called the *standard form of a polynomial*. The term that does not contain the variable is written at the end. It is known as the *constant term* because its value remains the same no matter what value is given to the variable. It is usual to denote a polynomial in the variable x by a symbol like $p(x)$, $q(x)$, $r(x)$, etc.

The highest exponent of the variable in a polynomial is called the degree of the polynomial.

Illustration 1 : The degree of the polynomial $5x^2 + 13x - 9$ in x is 2. We say that it is a *second-degree polynomial* or a *polynomial of degree* 2.

Illustration 2 : The degree of the polynomial $y^3 + 17y$ in y is 3. We say that it is a *third-degree polynomial* or a *polynomial of degree* 3.

Illustration 3 : The degree of the polynomial $2p + 3$ in p is 1.

Illustration 4 : A *constant* like 3 or -72 is known as a *polynomial of degree* 0, because we can think of a number like 3 as $3x^0$.

Remark : A second-degree polynomial is also called a *quadratic* polynomial. A third-degree polynomial is also called a *cubic* polynomial. A fourth-degree polynomial (e.g., $3x^4$ or $2x^4 + 3x^2 + 9x + 4$) is also called a *quartic* or a *biquadratic* polynomial.

7.3 Division of a Polynomial by a Polynomial

Recall the process of division in case of integers. We faced two different types of situations. Firstly, the case in which one integer was exactly divisible by the other. For

example, when we divided 12 by 4, we got a quotient 3. (Incidentally, this also meant that dividing 12 by 3 would produce a quotient 4.) This really meant that we were only simplifying $12 \div 4$ to the integer 3. The second type of situations involved integers where we could not divide exactly and a remainder was obtained. For example, we could not divide 12 by 7 exactly. Dividing 12 by 7, we got a quotient 1 and a remainder 5.

In case of polynomials in one variable also, we face similar situations so far as the process of division is concerned. Sometimes, it is just a case of simplification only. Dividing one polynomial by another produces a quotient polynomial and a remainder. *In some cases the remainder is zero, and in some non-zero.*

7.4 Division of a Polynomial by Polynomial : Zero-remainder

We begin with the simple case, where dividing a polynomial $p(x)$ by another polynomial $q(x)$ produces a third polynomial $r(x)$. Recall that the process of division is related with the process of multiplication in a specific way. At the most elementary level, every multiplication fact for integers gives rise to two division facts as illustrated below :

Multiplication fact : $5 \times 4 = 20$

Related division facts : $20 \div 5 = 4, 20 \div 4 = 5$

We may extend this idea for the division process of polynomials. If a given polynomial is the product of two other polynomials, then this multiplication fact for polynomials gives rise to two division facts for polynomials.

Throughout the rest of the Chapter, we assume that the polynomials in any given example involve the same single variable. Thus, we may try to divide $p^2 + 3p - 28$ by $p-4$ but not by $y-4$.

7.4.1 Division of a Monomial by a Monomial

Illustration 5 : Multiplication fact : $x^3 \times x^2 = x^5$

Related division facts : $x^5 \div x^3 = x^2$, $x^5 \div x^2 = x^3$

We may write these division facts as follows :

$$\frac{x^5}{x^3} = x^2, \quad \frac{x^5}{x^2} = x^3$$

Illustration 6 : Multiplication fact : $5x^4 \times 3x = 15x^5$

Related division facts : $15x^5 \div 5x^4 = 3x$, $15x^5 \div 3x = 5x^4$

We may write these division facts as follows :

$$\frac{15x^5}{5x^4} = 3x, \quad \frac{15x^5}{3x} = 5x^4$$

The above illustrations suggest that we may use the laws of exponents for numbers in case of literals also. In fact, we shall be justified in doing so because we know that literals, after all, stand for numbers only. This gives us the following two rules for dividing a monomial by a monomial.

Rule 1 : The coefficient of the quotient of two monomials is equal to the quotient of the coefficients of the monomials in question.

Rule 2 : The variable part in the quotient of two monomials is equal to the quotient of the variable parts in the monomials in question.

Example 1 : Divide: (i) $-20\,x^4$ by $10\,x$ (ii) $3y^3$ by $\sqrt{3}y$.

Solution : (i) $$\frac{-20x^4}{10x} = \left(\frac{-20}{10}\right)\left(\frac{x^4}{x}\right) = -2x^3$$

(ii) $$\frac{3y^3}{\sqrt{3}y} = \left(\frac{3}{\sqrt{3}}\right)\left(\frac{y^3}{y}\right) = \sqrt{3}y^2$$

7.4.2 Division of a Polynomial by a Monomial

There are two convenient methods for dividing a polynomial by a given monomial. In the first method, division is carried out in the following steps :

Step1 : Split the polynomial to be divided into separate terms.

Step 2 : Divide each term by the given monomial.

Step 3 : Add the quotients.

Let us illustrate this method by an example.

Example 2 : Divide $3y^3 + 15y^2 + 12y$ by $3y$.

Solution: Step 1 : The given polynomial has three terms, namely $3y^3$, $15y^2$ and $12y$.

Step 2 : Dividing each term by the given monomial $3y$, we get

$$\frac{3y^3}{3y}, \frac{15y^2}{3y}, \frac{12y}{3y}$$

or $$y^2, 5y, 4.$$

Step 3 : Adding the above quotients, we get $y^2 + 5y + 4$ as the desired result. Hence,

$$\frac{3y^3 + 15y^2 + 12y}{3y} = y^2 + 5y + 4$$

Remark : Once you understand the process, you may shorten the working as shown in the example that follows.

Example 3 : Divide $34x^3 - 17x^2 + 51x$ by $17x$.

Solution :
$$\frac{34x^3 - 17x^2 + 51x}{17x} = \frac{34x^3}{17x} + \frac{-17x^2}{17x} + \frac{51x}{17x}$$
$$= 2x^2 - x + 3$$

Second Method : We already know what is meant by a factor of an algebraic expression. Since a polynomial is a special type of algebraic expression only, we know what is meant by a factor of a polynomial. We have also learnt how to factorise algebraic expressions. So we factorise the polynomial to be divided by a monomial in such a way that one of the factors is the given monomial. After that, the process of division may be carried out more conveniently as shown in the example below.

Example 4 : Divide $4q^3 - 10q^2 + 5q$ by $2q$.

Solution : $4q^3 - 10q^2 + 5q = 2q \times 2q^2 - 2q \times 5q + 2q \times \left(\frac{5}{2}\right)$

$$= 2q \times \left(2q^2 - 5q + \frac{5}{2}\right)$$

Hence, $4q^3 - 10q^2 + 5q = 2q \times \left(2q^2 - 5q + \frac{5}{2}\right)$, so that

$$\frac{4q^3 - 10q^2 + 5q}{2q} = \frac{2q\left(2q^2 - 5q + \frac{5}{2}\right)}{2q}$$

$$= 2q^2 - 5q + \frac{5}{2}$$

We cancelled out the common factor $2q$ from the numerator and the denominator to get the desired answer.

7.4.3 Division of a Polynomial by a Binomial : Factor Method

We now extend the method of Example 4 to divide a polynomial by a binomial. Recall that we are still discussing the zero-remainder case. As in Example 4 above, if possible, we factorise the polynomial to be divided so that one of the factors is equal to the binomial by which we wish to divide. Then, we cancel out the common factor and get the answer.

Example 5 : Divide $a^2 + 4a - 5$ by $a - 1$.

Solution : $a^2 + 4a - 5 = (a + 5)(a - 1)$, using Identity I of Chapter 6.

Therefore,
$$\frac{a^2 + 4a - 5}{a - 1} = \frac{(a + 5)(a - 1)}{a - 1} \qquad (a \neq 1)$$
$$= a + 5$$

We cancelled out the common factor $(a - 1)$ from the numerator and the denominator to get the desired answer.

Remark : While writing a polynomial in a general form, we often use x for the variable and a, b etc. for constants. Sometimes, we use a as a variable, if there is no confusion.

7.4.4 Division of a Polynomial by a Binomial : Long Division Method

As you must have realised it will not always be possible for you to factorise the polynomial to be divided. We shall now explain a method by which you can always divide a polynomial by a given binomial. This method is known as the *method of long division*. We shall illustrate the method with the help of examples.

Recall that in case of integers, the integer to be divided is called the dividend, and the number by which we divide is called the divisor. We shall use the same terms in case of polynomials also. The terms quotient and remainder will also have similar meanings.

Example 6 : Divide $12 - 14x^2 - 13x$ by $3 + 2x$.

Solution : We carry out the process of division by means of the following steps :

Step 1 : We write the dividend $(12 - 14x^2 - 13x)$ and the divisor $(3 + 2x)$ in the standard form. This gives

Dividend : $-14x^2 - 13x + 12$, Divisor : $2x + 3$.

Step 2 : We divide the first term of the dividend by the first term of the divisor. That is, we divide $-14x^2$ by $2x$, and obtain $-7x$. This gives us the first term of the quotient : $-7x$.

$\dfrac{-14x^2}{2x} = -7x$	$\begin{array}{r} -7x \\ 2x\overline{)-14x^2} \\ \mp 14x^2 \\ \hline 0 \end{array}$	Quotient $= -7x$

Step 3 : We multiply the divisor by the first term of the quotient and subtract the product from the dividend. That is, we multiply $2x + 3$ by $-7x$ and subtract the product $-14x^2 - 21x$ from the dividend $-14x^2 - 13x + 12$. This gives us a remainder $8x + 12$.

$$(2x + 3) \times (-7x) = -14x^2 - 21x \quad \begin{array}{r} -14x^2 - 13x + 12 \\ \underset{+}{-}14x^2 \underset{+}{-} 21x \\ \hline 8x + 12 \end{array}$$

Step 4 : We treat the above remainder $8x + 12$ as the new dividend. The divisor remains the same. We repeat Step 2 to get the next term of the quotient. That is, we divide the first term $(8x)$ of the dividend by the first term $(2x)$ of the divisor and obtain 4. Thus, 4 is the second term in the quotient.

$$\frac{8x}{2x} = 4 \quad \begin{array}{c} \text{Quotient} \\ -7x + 4 \end{array}$$

Step 5 : We multiply the divisor by the term of the quotient just obtained and subtract the product from the dividend. That is, we multiply $2x + 3$ by 4 and subtract the product $8x + 12$ from the dividend $8x + 12$. This gives us 0 as the remainder.

$$(2x + 3) \times 4 = 8x + 12 \quad \begin{array}{r} 8x + 12 \\ \underset{-}{8x} \underset{-}{+} 12 \\ \hline 0 \end{array}$$

Step 6 : Thus, the quotient in full is $-7x + 4$ and the remainder is zero. Hence, we say that

$$(-14x^2 - 13x + 12) \div (2x + 3) = -7x + 4$$

The above process is displayed as follows :

$$\begin{array}{r|l} & -7x + 4 \\ \hline 2x+3 & -14x^2 - 13x + 12 \\ & \underset{+}{-}14x^2 \underset{+}{-} 21x \\ \hline & \qquad\quad 8x + 12 \\ & \qquad\quad \underset{-}{8x} \underset{-}{+}12 \\ \hline & \qquad\qquad 0 \end{array}$$

Notice that $-14x^2 - 13x + 12 = (2x + 3)(-7x + 4)$ (1)

i.e., Dividend = Divisor × Quotient

Thus, we see from relation (1) that both of $(2x + 3)$ and $(-7x + 4)$ are factors of $-14x^2 - 13x + 12$. In other words, *both the divisor and the quotient in this example are factors of the dividend.*

Let us take another example.

Example 7 : Divide $2 + 7x + 7x^2 + 2x^3$ by $1 + 2x$.

Solution : We carry out the process of division by means of the following steps :

Step 1 : We write the dividend $(2 + 7x + 7x^2 + 2x^3)$ and the divisor $(1 + 2x)$ in the standard form. This gives

Dividend : $2x^3 + 7x^2 + 7x + 2$, Divisor : $2x + 1$.

Step 2 : We divide the first term of the dividend by the first term of the divisor. That is, we divide $2x^3$ by $2x$ and obtain x^2. This gives us the first term of the quotient : x^2.

$$\frac{2x^3}{2x} = x^2 \qquad \text{Quotient} \quad x^2$$

Step 3 : We multiply the divisor by the first term of the quotient and subtract the product from the dividend. That is, we multiply $2x + 1$ by x^2 and subtract the product $2x^3 + x^2$ from the dividend $2x^3 + 7x^2 + 7x + 2$. This gives us a remainder $6x^2 + 7x + 2$.

$$\begin{array}{l|l} (2x+1)\times x^2 & 2x^3+7x^2+7x+2 \\ = 2x^3+x^2 & \underline{\underset{-}{2x^3} \underset{-}{+} x^2} \\ & \quad 6x^2+7x+2 \end{array}$$

Step 4 : We treat the above remainder $6x^2 + 7x + 2$ as the new dividend. The divisor remains the same. We repeat Step 2 to get the next term of the quotient. That is, we divide the first term ($6x^2$) of the dividend by the first term ($2x$) of the divisor and obtain $3x$. Thus, $3x$ is the second term in the quotient.

$$\frac{6x^2}{2x} = 3x \qquad \text{Quotient} \quad x^2+3x$$

Step 5 : We multiply the divisor by the term of the quotient just obtained and subtract the product from the dividend. That is, we multiply $2x + 1$ by $3x$ and subtract the product $6x^2 + 3x$ from the dividend $6x^2 + 7x + 2$. This gives us $4x + 2$ as the remainder.

$$\begin{array}{l|l} (2x+1)\times 3x & 6x^2+7x+2 \\ = 6x^2+3x & \underline{\underset{-}{6x^2} \underset{-}{+} 3x} \\ & \qquad 4x+2 \end{array}$$

Step 6 : We treat the above remainder $4x + 2$ as the new dividend. The divisor remains the same. We repeat Step 2 to get the next term of the quotient. That is, we divide the first term ($4x$) of the dividend by the first term ($2x$) of the divisor and obtain 2 as the next term in the quotient.

$$\frac{4x}{2x} = 2 \qquad \text{Quotient} \quad x^2+3x+2$$

Step 7 : We multiply the divisor by the term of the quotient just obtained and subtract the product from the dividend. That is, we multiply $2x + 1$ by 2 and subtract the product $4x + 2$ from the dividend $4x + 2$. This gives us 0 as the remainder.

$$\begin{array}{l|l} (2x+1)\times 2 & 4x+2 \\ = 4x+2 & \underline{\underset{-}{4x} \underset{-}{+} 2} \\ & \qquad 0 \end{array}$$

Step 8 : Thus, the quotient in full is $x^2 + 3x + 2$ and the remainder is zero. Hence, we say that

$$(2x^3 + 7x^2 + 7x + 2) \div (2x + 1) = x^2 + 3x + 2$$

The above process is displayed as follows :

$$\begin{array}{r|l} & x^2+3x+2 \\ \hline 2x+1 & 2x^3+7x^2+7x+2 \\ & \underline{\underset{-}{2x^3} \underset{-}{+} x^2} \\ & \quad 6x^2+7x+2 \\ & \quad \underline{\underset{-}{6x^2} \underset{-}{+} 3x} \\ & \qquad\quad 4x+2 \\ & \qquad\quad \underline{\underset{-}{4x} \underset{-}{+} 2} \\ & \qquad\qquad 0 \end{array}$$

Notice that $(2x^3 + 7x^2 + 7x + 2) = (2x + 1)(x^2 + 3x + 2)$ (1)

i.e., Dividend = Divisor × Quotient

Thus, we see from relation (1) that both of $(2x + 1)$ and $(x^2 + 3x + 2)$ are factors of $2x^3 + 7x^2 + 7x + 2$. In other words, *both the divisor and the quotient in this example also are factors of the dividend.*

The above two examples suggest that the following result, which holds for integers, also holds for polynomials :

If an integer m on being divided by a non-zero integer n leaves a remainder 0 and gives a quotient q, then m = nq. Thus, n is a factor of m.

In other words, *if the remainder is 0, then Dividend = Divisor × Quotient*. Hence, in case of polynomials, we have :

If a polynomial $f(x)$ on being divided by a non-zero polynomial $g(x)$ leaves a remainder 0 and gives a quotient $q(x)$, then $f(x) = g(x)\,q(x)$. Thus, $g(x)$ is a factor of $f(x)$.

In other words, in case of polynomials also, we have :

If the remainder is 0, then Dividend = Divisor × Quotient.

Remarks 1 : The above relation between dividend, divisor and quotient proves to be very useful in deciding whether a given polynomial is a factor of some polynomial.

2. Not only the divisor, but also the quotient is a factor of the dividend if the remainder is zero.

3. There is no need to write the steps in detail. These have been written so that you understand the process. We carry out the division as shown in the next example.

Example 8 : Find out whether or not $6x + 5$ is a factor of $6x^2 - 7x - 10$.

Solution : We divide the polynomial $6x^2 - 7x - 10$ by $6x + 5$.

$$
\begin{array}{r|l}
 & \quad x - 2 \\
\hline
6x+5 & 6x^2 - 7x - 10 \\
 & \underline{\overset{-}{}6x^2 \underset{-}{+} 5x} \\
 & \quad -12x - 10 \\
 & \quad \underline{\underset{+}{-}12x \underset{+}{-} 10} \\
 & \qquad\qquad 0
\end{array}
$$

Since the remainder is zero, we find that $6x + 5$ is a factor of $6x^2 - 7x - 10$.

EXERCISE 7.1

1. Which of the following expressions are not polynomials?

(i) $3z^3 - \sqrt{5}z + 9$ (ii) $3\sqrt{z} + 4z + 5z^2$

(iii) $\sqrt{a}x + x^2 - x^3$ (iv) $\sqrt{a}\,x^{\frac{1}{2}} + ax + 9x^2 + 5$

(v) $2x^{-2} + 3x^{-1} + 4 + 5x$ (vi) $x^2 + x^{-2}$

Write each of the following polynomials in the standard form . Also write the degree of each:

2. $y^2 + 6y + 9 + 4y^4$ **3.** $4q^2 - 13q^5 + 12q$

4. $\left(z+\frac{3}{4}\right)\left(z+\frac{4}{3}\right)$ **5.** $(x^2 + 4)(x^2 + 9)$

6. $y^2 + 12 - 5y^8$ **7.** $q^2 + 4q^8 - q^6$

8. $p^2 + 16 + p^7$ **9.** $y^2 + y^3 - \frac{5}{7}y^{11}$

10. $(z^3 - 14)(z^3 - 1)$ **11.** $(z^3 - 1)(z^3 - 8)$

12. $(y^3 - 2)(y^3 + 11)$ **13.** $\left(x^3 - \frac{3}{8}\right)\left(x^3 + \frac{16}{17}\right)$

Divide :

14. $2x^2$ by $2x$ **15.** $-3x^3$ by x^2

16. $\frac{2}{3}x^2$ by x **17.** $\sqrt{5}x^4$ by $5x^3$

18. $\sqrt{3}\,a^3$ by $2a$ **19.** $4a^4$ by $-2\sqrt{2}a^2$

20. $x + 2x^2 + 3x^3$ by $2x$ **21.** $y^4 - 3y^3 + \frac{1}{2}y^2$ by $3y$

22. $-4p^3 + 4p^2 + p + 4$ by $2p$ **23.** $-x^4 + x^2$ by $\sqrt{2}\,x^2$

24. $5z^3 - 6z^2 + 7z$ by $2z$ **25.** $\sqrt{3}\,q^4 + 2\sqrt{3}\,q^3$ by $3q^2$

26. Divide by long division method. Verify your result by factor method.

(i) $x^2 + 6x + 8$ by $x + 4$ (ii) $x^2 + 7x + 10$ by $x + 5$

(iii) $y^2 - y - 12$ by $y - 4$ (iv) $y^2 - 5y + 6$ by $y - 2$

(v) $z^2 - 8z + 15$ by $z - 5$ (vi) $x^4 + 3x^2 + 2$ by $x^2 + 2$

[*Hint :* Write $x^2 = y$.]

27. Verify that the given binomial is a factor of the given polynomial by showing that the relevant remainder is zero :

(i) $2x + 3,\ 2x^2 + 5x + 3$ (ii) $2x + 1,\ 6x^2 + x - 1$

(iii) $2y - 1,\ 8y^2 - 2y - 1$ (iv) $5a + 3,\ 10a^2 - 9a - 9$

(v) $3b - 1,\ -3b^2 + 13b - 4$ (vi) $p^2 + 3,\ 4p^4 + 7p^2 - 15$

7.5 Division of a Polynomial by a Polynomial : Non-zero Remainder

So far, we have considered the case where a polynomial being divided by a monomial or a binomial leaves no remainder (i.e., zero remainder). Like numbers, we say that the monomial or binomial in question divides the polynomial completely or exactly. Now we consider the case where the remainder is not zero.

Recall that in case of numbers, we continue the division till we obtain a remainder that is less than the divisor. In case of polynomials, we do not have the concept of one polynomial being less than another. Instead, we compare the degrees of the divisor and the remainder. Since the degrees of polynomials are integers, we can compare the same. We continue the process of division in case of polynomials till we obtain a remainder with degree less than that of the divisor. We illustrate the process through an example.

Example 9 : Divide the polynomial $5x\,(x^2 - x + 1) - (9 + 4x^4)$ by $4x - 1$.

Solution : The given polynomial is not in the standard form. Let us first write it in the standard form.

$$
\begin{aligned}
5x\,(x^2 - x + 1) - (9 + 4x^4) &= 5x^3 - 5x^2 + 5x - 9 - 4x^4 \\
&= -4x^4 + 5x^3 - 5x^2 + 5x - 9
\end{aligned}
$$

We now carry out the division as explained in the previous section.

$$
\begin{array}{r r l}
 & -x^3 + x^2 - x + 1 & \\
4x - 1 \,\big) & -4x^4 + 5x^3 - 5x^2 + 5x - 9 & \\
 & -4x^4 + x^3 \qquad\qquad\qquad\quad & \\
 & + \quad\ - \qquad\qquad\qquad\qquad & \\
\hline
 & 4x^3 - 5x^2 + 5x - 9 & \text{(Degree of remainder not less than that of divisor; continue division)} \\
 & 4x^3 - x^2 \qquad\qquad\ & \\
 & - \quad\ + \qquad\qquad\quad & \\
\hline
 & -4x^2 + 5x - 9 & \text{(Degree of remainder not less than that of divisor; continue division)} \\
 & -4x^2 + x \qquad\ & \\
 & + \quad\ - \qquad\quad & \\
\hline
 & 4x - 9 & \text{(Degree of remainder not less than that of divisor; continue division)} \\
 & 4x - 1 & \\
 & - \quad + & \\
\hline
 & -8 & \text{(Degree of remainder less than that of divisor; stop division)}
\end{array}
$$

Thus, the quotient is $-x^3 + x^2 - x + 1$ and the remainder is -8.

In the above example, observe that

$$-4x^4 + 5x^3 - 5x^2 + 5x - 9 = (4x - 1) \times (-x^3 + x^2 - x + 1) + (-8)$$

In other words,

$$\textit{Dividend} = \textit{Divisor} \times \textit{Quotient} + \textit{Remainder}$$

Example 10 : Divide the polynomial $3y^4 - y^3 + 12y^2 + 2$ by $3y^2 - 1$.

Solution : We carry out the division as explained in the previous section.

$$\begin{array}{r|l}
 & y^2 - \frac{1}{3}y + \frac{13}{3} \\ \hline
3y^2 - 1 & 3y^4 - y^3 + 12y^2 \qquad + 2 \\
 & \underline{\mathop{3y^4}_{-} \qquad \mathop{-y^2}_{+}} \\
 & -y^3 + 13y^2 \qquad + 2 \\
 & \underline{\mathop{-y^3}_{+} \qquad \mathop{+\frac{1}{3}y}_{-}} \\
 & 13y^2 - \frac{1}{3}y + 2 \\
 & \underline{\mathop{13y^2}_{-} \qquad \mathop{-\frac{13}{3}}_{+}} \\
 & -\frac{1}{3}y + \frac{19}{3}
\end{array}$$

(Leaving a blank space for the missing term in y. It is important to write like terms one below the other even if we have to leave blank spaces.)

(Degree of remainder not less than that of divisor, continue division.)

(Degree of remainder not less than that of divisor, continue division.)

(Degree of remainder less than that of divisor, stop division.)

Thus, we find that when we divide $3y^4 - y^3 + 12y^2 + 2$ by $3y^2 - 1$, we get $y^2 - \frac{1}{3}y + \frac{13}{3}$ as the quotient and $-\frac{1}{3}y + \frac{19}{3}$ as the remainder.

Observe that

$$3y^4 - y^3 + 12y^2 + 2 = (3y^2 - 1) \times \left(y^2 - \frac{1}{3}y + \frac{13}{3}\right) + \left(-\frac{1}{3}y + \frac{19}{3}\right)$$

In other words, in this example also,

$$\textit{Dividend} = \textit{Divisor} \times \textit{Quotient} + \textit{Remainder}$$

In fact, this result is always true. The reason is that if we divide (Dividend – Remainder)

by the Divisor, then the remainder would be zero. Hence, as in the previous section, we shall have

$$\text{Dividend} - \text{Remainder} = \text{Divisor} \times \text{Quotient}$$

or

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Remark : The above relation helps us in deciding whether or not one of the two given polynomials is a factor of the other. If the remainder is not zero, then neither the divisor nor the quotient is a factor of the dividend.

Let us record the important facts about long division of polynomials :

1. Before proceeding to divide, you must write the dividend and the divisor in the standard form. That is, arrange the terms of both the dividend and the divisor in decreasing order of exponents of the variable.
2. During the process of divison, write like terms one below the other. To do so, you have to leave gaps for the missing terms at every step.
3. Keep on dividing so long as the degree of the remainder obtained is not less than that of the divisor.
4. Stop dividing when a remainder is obtained with a degree less than that of the divisor. This is the required remainder.
5. If the remainder is zero, then the divisor is a factor of the dividend. Therefore, to examine whether a given polynomial $f(x)$ is a factor of another polynomial $g(x)$, we divide $g(x)$ by $f(x)$ and examine the remainder. If it is zero, then $f(x)$ is a factor of $g(x)$, otherwise not.

7.5.1 A Quick and Short Version of Long Division

We give an alternate method for *long* division that is very *short*. Suppose we wish to divide $x^3 + 4x^2 - 3x - 7$ by $x - 3$. Write the polynomial to be divided in the top line. Leave one line blank. Draw a line. Below the line, write the divisor three times leaving blank space for a numerical coefficient as shown on the side.

$$\begin{array}{ccc} x^3 \quad + 4x^2 & -\ 3x & -\ 7 \\ & & \\ \hline (x-3) & (x-3) & (x-3) \end{array}$$

As in the usual method, we calculate the first term of the quotient which is x^2 here. Multiply the first '$(x-3)$' in the bottom line by x^2 and write the product in the blank line. The term in x^3 is taken care of. To

$$\begin{array}{ccc} x^3 \quad + 4x^2 & -\ 3x & -\ 7 \\ (x^3 - 3x^2) + & (7x^2 & \\ \hline \underline{\underline{x^2}}\,(x-3) & (x-3) & (x-3) \end{array}$$

get the correct term in x^2, we write $7x^2$ in the middle line (so that together with $-3x^2$, it gives $4x^2$ as in the dividend). From $7x^2$, we get the second term $7x$ of the quotient. Write $+ 7x$ before the second $(x-3)$ in the bottom line. Write the product $(7x^2 - 21x)$ in the middle line completing the entry '$(7x^2$'.

$$\begin{array}{l} x^3 \quad + 4x^2 - 3x - 7 \\ (x^3 - 3x^2) + (7x^2 - 21x) \\ \hline \underline{\underline{x^2}}\,(x-3) + \underline{\underline{7x}}\,(x-3) \quad (x-3) \end{array}$$

Now adjust the x-term by writing $(18\,x$ in the middle line. This gives us the next term 18 of the quotient.

$$\begin{array}{l} x^3 \quad + 4x^2 - 3x - 7 \\ (x^3 - 3x^2) + (7x^2 - 21x) + (18x \\ \hline \underline{\underline{x^2}}\,(x-3) + \underline{\underline{7x}}\,(x-3) \qquad (x-3) \end{array}$$

Multiply the last $(x-3)$ in the bottom line by 18 and write the product in the middle line. Finally, adjust the constant term by writing 47 in the middle line. This term 47 is the remainder. The final result is shown on the right.

$$\begin{array}{l} x^3 \quad + 4x^2 - 3x - 7 \\ (x^3 - 3x^2) + (7x^2 - 21x) + (18x - 54) + 47 \\ \hline \underline{\underline{x^2}}\,(x-3) + \underline{\underline{7x}}\,(x-3) + \underline{\underline{18}}\,(x-3) \end{array}$$

Thus, the quotient is $x^2 + 7x + 18$ (shown doubly underlined). Further, remainder is 47.

EXERCISE 7.2

Divide the first polynomial by the second. Write the quotient and the remainder.

1. $3a^2 + 5a + 7,\ a + 2$

2. $10b^2 - 7b + 8,\ 5b - 3$

3. $6p^3 + 5p^2 + 4,\ 2p + 1$

4. $8q^3 - 6q^2 + 4q - 1,\ 4q + 2$

5. $12x^3 - 8x^2 - 6x + 10,\ 3x - 2$

6. $16x^4 + 12x^3 - 10x^2 + 8x + 20,\ 4x - 3$

7. $5y^3 - 6y^2 + 6y - 1,\ 5y - 1$

8. $z^4 + z^3 + z^2,\ z + 1$

9. $x^4 - x^3 + 5x,\ x - 1$

10. $y^4 + y^2,\ y^2 - 2$

11. Verify the result *Dividend = Divisor × Quotient + Remainder* for all the questions above.

What can be the possible degrees of the remainder when we divide

12. $x^4 + x^3$ by $x + 9$?

13. $x^2 + x + 1$ by $x - 2$?

14. $x^4 + 10x^3 - 9$ by $x^3 + 4$?

15. $y^4 + y^2 - y - 3$ by $y^2 + 6$?

Find, whether or not the first polynomial is a factor of the second:

16. $x + 1,\ 2x^2 + 5x + 4$

17. $3x - 1,\ 6x^2 + x - 1$

18. $4y + 1, 8y^2 - 2y + 1$

19. $2a - 3, 10a^2 - 9a - 5$

20. $4 - z, 3z^2 - 13z + 4$

21. $4z^2 - 5, 4z^4 + 7z^2 + 15$

22. $y - 2, 3y^3 + 5y^2 + 5y + 2$

Things to Remember

1. Algebraic expressions in which the variables involved have only non-negative integral exponents are called polynomials.
2. A polynomial that involves only one variable is called a polynomial in one variable.
3. The highest exponent of the variable in various terms of a polynomial in one variable is called its degree.
4. A constant is a polynomial of degree zero.
5. The standard form of a polynomial in one variable is that in which the terms of the polynomial are written in the decreasing order of the exponents of the variable.
6. The coefficient in the quotient of two monomials is equal to the quotient of their coefficients.
7. The variable part in the quotient of two monomials is equal to the quotient of the variable parts in the given monomials.
8. If on dividing a polynomial (dividend) by a polynomial (divisor), a zero remainder is obtained, then the divisor is a factor of the dividend. In such cases, quotient is also a factor of the dividend. Further,

   $$\text{Dividend} = \text{Divisor} \times \text{Quotient}$$

9. In general,

   $$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

10. The degree of the remainder is always less than the degree of the divisor.
11. Before performing long division, the divisor and the dividend must be written in the standard form.
12. While performing long division, like terms are written one below the other, leaving gaps wherever necessary.

CHAPTER

8

EQUATIONS IN ONE VARIABLE

8.1 Introduction

You are familiar with linear equations in one variable. Such an equation is of the form $ax + b = c$, where a, b and c are numbers, $a \neq 0$ and x is the variable. A value of the variable that satisfies the equation is known as a solution or root of the equation. Recall that the equality sign of an equation does not change, if we

- add the same number to both the sides of the equation.
- subtract the same number from both the sides of the equation.
- multiply or divide both sides of the equation by the same non-zero number.
- transpose a term from one side of the equation to the other.

You know that we can solve some real life problems by converting the same to linear equations and then solving these equations. This activity would be continued in this Chapter.

In this Chapter, we shall also learn to solve equations of the type

$\dfrac{ax+b}{cx+d} = k$, where a, b, c, d and k are numbers, and $cx + d \neq 0$, by reducing them to linear equations. We shall then solve some word problems by converting the same to equations of the above type.

8.2 Equations of the Form $\dfrac{ax+b}{cx+d} = k$

Let us try to find two numbers, in the ratio 7 : 8, whose sum is 45. To find such numbers, let us suppose that the smaller of the two numbers is x. Since the sum of these numbers is 45, the second number must be $45 - x$. Since the numbers are in the ratio 7 : 8, we must have

$$\frac{x}{45-x} = \frac{7}{8}$$

This gives us an equation of the type

$$\frac{ax+b}{cx+d} = k, \text{ where } a = 1, b = 0, c = -1, d = 45, \text{ and } k = \frac{7}{8}.$$

Thus, in order to solve the above problem, we need to solve an equation of the type $\frac{ax+b}{cx+d} = k$. This equation does not appear to be linear but it is a linear equation in disguise. Let us illustrate this by means of some examples.

Example 1: Solve the equation $\frac{3x+8}{2x+7} = 4$.

Solution : Observe that if $2x+7$ were not present in the denominator on the LHS, the given equation would have been a linear equation. So, we shall try to get rid of this expression. Recall that the literals stand for some numbers only. Thus x, and therefore, $2x+7, 3x+8$ and $\frac{3x+8}{2x+7}$ also represent numbers. Hence, we may multiply both sides of the given equation $\frac{3x+8}{2x+7} = 4$ by $2x+7$ without affecting the equality sign. This gives

$$\frac{3x+8}{2x+7} \times (2x+7) = 4(2x+7)$$

or $\quad 3x + 8 = 8x + 28$

or $\quad 3x - 8x = 28 - 8$ (Transposing $8x$ to LHS, 8 to RHS)

or $\quad -5x = 20$

or $\quad x = -4$

Check : When $x = -4$,

$$\text{LHS} = \frac{3(-4)+8}{2(-4)+7} = \frac{-4}{-1} = 4 = \text{RHS}$$

Hence, the solution is correct.

Example 2 : Solve the equation $\frac{5x+2}{2x+3} = \frac{12}{7}$.

Solution : As in the previous example, let us multiply both sides of the given equation by $2x+3$. This gives

$$\frac{5x+2}{2x+3} \times (2x+3) = \frac{12}{7} \times (2x+3)$$

or $\quad 5x + 2 = \frac{12}{7}(2x+3) \qquad (1)$

or $$5x + 2 = \frac{24}{7}x + \frac{36}{7}$$

or $$5x - \frac{24}{7}x = \frac{36}{7} - 2 \qquad \left[\text{Transposing } \frac{24}{7}x \text{ and } 2\right]$$

or $$\frac{11}{7}x = \frac{22}{7}$$

or $$x = \frac{22}{7} \times \frac{7}{11} \qquad \left[\text{Multiplying both sides by } \frac{7}{11}\right]$$

or $$x = 2$$

Check : At $x = 2$,

$$\text{LHS} = \frac{5x+2}{2x+3} = \frac{5 \times 2 + 2}{2 \times 2 + 3} = \frac{12}{7} = \text{RHS}$$

Hence, the solution is correct.

Remark : Just as we got rid of the denominator $2x + 3$ on the LHS in the above example, we could get rid of the denominator 7 also on the RHS. If we do so, we shall not have to work with fractions. Let us modify the above solution. Multiplying both sides of equation (1) by 7, we get

$$(5x + 2) \times 7 = \frac{12(2x + 3)}{7} \times 7$$

or $$(5x + 2) \times 7 = 12(2x + 3) \qquad (2)$$

or $$35x + 14 = 24x + 36$$

or $$11x = 22$$

or $$x = 2, \text{ as before.}$$

We could have done even better. Instead of multiplying by $2x + 3$ and 7 in stages, we could have multiplied the two sides of the given equation by $7(2x + 3)$ all at once. This would have given

$$\frac{5x+2}{2x+3} \times 7(2x + 3) = \frac{12}{7} \times 7(2x + 3)$$

or $$7(5x + 2) = 12(2x + 3) \qquad (3)$$

This is the same as equation (2). Thus, the solution of the equation becomes simpler.

Now look at the given equation and equation (3) carefully :

Given Equation	*Simplified Form of the Given Equation*
$\dfrac{5x+2}{2x+3} = \dfrac{12}{7}$	$7 \times (5x+2) = 12 \times (2x+3)$

What do you notice? All we have done is to :

(i) multiply the numerator of the LHS by the denominator of the RHS,

$$\frac{5x+2}{2x+3} = \frac{12}{7}$$

(ii) multiply the numerator of the RHS by the denominator of the LHS,

$$\frac{5x+2}{2x+3} = \frac{12}{7}$$

(iii) equate the expressions obtained in (i) and (ii). $\quad 7 \times (5x+2) = 12 \times (2x+3)$

For obvious reasons, we call this method of solution, the *method of cross multiplication.* Let us now illustrate the method of cross multiplication by examples.

Example 3 : Solve the equation $\dfrac{x+7}{3x+16} = \dfrac{4}{7}$.

Solution : Cross multiplying, we get

$$\frac{x+7}{3x+16} = \frac{4}{7}$$

$$7 \times (x+7) = 4 \times (3x+16)$$

or $\quad 7x + 49 = 12x + 64$

or $\quad 7x - 12x = 64 - 49$

or $\quad -5x = 15$

or $\quad x = -3$

Check : At $x = -3$,

$$\text{LHS} = \frac{x+7}{3x+16} = \frac{-3+7}{3\times(-3)+16} = \frac{4}{7} = \text{RHS}$$

Hence, the solution is correct.

Example 4 : Solve the equation $\dfrac{4x+1}{8x-4} = 2$.

Solution : $\dfrac{4x+1}{8x-4} = 2$

or $\quad \dfrac{4x+1}{8x-4} = \dfrac{2}{1} \quad \left(\text{Treating the integer 2 as the rational number } \dfrac{2}{1}\right)$

Cross multiplying, we get

$$1 \times (4x + 1) = 2 \times (8x - 4)$$

or $\quad 4x + 1 = 16x - 8$

or $\quad 1 + 8 = 16x - 4x$ (Transposing – 8 to LHS and $4x$ to RHS)

or $\quad 9 = 12x$

or $\quad x = \frac{9}{12} = \frac{3}{4}$

Check : At $x = \frac{3}{4}$,

$$\text{LHS} = \frac{4x+1}{8x-4} = \frac{4 \times \frac{3}{4} + 1}{8 \times \frac{3}{4} - 4} = \frac{3+1}{6-4} = \frac{4}{2} = 2 = \text{RHS}$$

Hence, the solution is correct.

Example 5 : Find a positive value of x which satisfies the equation $\frac{x^2+1}{x^2-1} = \frac{5}{4}$.

Solution : Let us write $x^2 = y$. Then the given equation becomes

$$\frac{y+1}{y-1} = \frac{5}{4}$$

Cross multiplying,

$$4(y+1) = 5(y-1)$$

or $\quad 4y + 4 = 5y - 5$

or $\quad 5 + 4 = 5y - 4y$ (Collecting like terms on either side)

$\therefore \quad y = 9$

Since $y = x^2$, we have

$$x^2 = 9 = 3^2 = (-3)^2$$

Taking the positive value, we get

$$x = 3$$

Let us examine if $x = 3$ satisfies the given equation. On checking, we find that $x = 3$ satisfies the given equation. Hence, 3 is the required value of x.

EXERCISE 8.1

Solve the following equations and check your solutions :

1. $\dfrac{5x-7}{3x} = 2$

2. $\dfrac{4x+18}{5x} = 2$

3. $\dfrac{4x}{2x+7} = 3$

4. $\dfrac{9x}{7-6x} = 15$

5. $\dfrac{2-z}{z+16} = \dfrac{3}{5}$

6. $\dfrac{2y+3}{y-9} = \dfrac{2}{7}$

7. $\dfrac{2y-9}{3y+4} = -1$

8. $\dfrac{5z-11}{3z+7} = -2$

9. $\dfrac{2y-4}{3y+2} = -\dfrac{2}{3}$

[*Hint* : $-\dfrac{2}{3} = \dfrac{-2}{3}$]

10. $\dfrac{5-7y}{2+4y} = -\dfrac{8}{7}$

11. $\dfrac{2k-5}{5k+2} = \dfrac{3}{22}$

12. $\dfrac{8p-5}{7p+1} = -\dfrac{5}{4}$

13. $\dfrac{\frac{2}{3}x+1}{x+\frac{1}{4}} = \dfrac{5}{3}$

14. $\dfrac{2x-\frac{3}{4}}{9x+\frac{4}{7}} = \dfrac{1}{4}$

15. $\dfrac{\frac{3}{4}y+7}{\frac{2}{5}y-4} = \dfrac{5}{4}$

16. $\dfrac{\frac{z}{4}-\frac{3}{5}}{\frac{4}{3}-7z} = -\dfrac{3}{20}$

17. $\dfrac{(2x+3)-(5x-7)}{6x+11} = -\dfrac{8}{3}$

18. $\dfrac{(y+1)-(2y+4)}{3-5y} = \dfrac{1}{23}$

19. $\dfrac{x^2-(x+1)(x+2)}{5x+1} = 6$

20. $\dfrac{(x+2)(2x-3)-2x^2+6}{x-5} = 2$

Find a positive value of variable x or y for which the given equation is satisfied :

21. $\dfrac{x^2-9}{5+x^2} = \dfrac{-5}{9}$

22. $\dfrac{y^2+4}{3y^2+7} = \dfrac{1}{2}$

8.3 Applications of Linear Equations

Recall that many problems involve relations among some known and some unknown quantities or numbers. In the previous Class, you learnt how to convert such

problems into linear equations in some cases. The solutions of these equations provide solutions of the corresponding practical problems. The important steps in the solution of these word problems are listed below :

1. Read the problem carefully and note down (i) what is given, and (ii) what is required.
2. Denote the unknown quantity by a literal x, y, z, u, v, w etc.
3. Translate the statements of the problem step by step into mathematical statements, to the extent possible.
4. Look for the quantities that are equal. Write the equations corresponding to these equality relations.
5. Solve the equations written in Step 4 above.
6. Check the solution by substituting the value of the unknown found in Step 5 above into the statements of the problem.

Let us now illustrate the above procedure by examples.

Example 6 : The sum of three consecutive multiples of 7 is 777. Find these multiples.

Solution : We use a variable to denote the quantity that is to be found. Here, we are required to find three numbers. But if one of these numbers, say the first one is known, then the other two can be obtained by adding 7 and 14 to it. (It is given that the numbers are consecutive multiples of 7.)

Let us assume that the first number is x. Then the other two numbers are $x + 7$ and $x + 14$.

We are given that the sum of these three consecutive multiples is 777. Hence,

$$x + (x + 7) + (x + 14) = 777$$

or $$3x + 21 = 777$$

or $$3x = 777 - 21 \quad \text{[Transposing 21]}$$

or $$3x = 756$$

or $$x = 252$$

Hence, the three consecutive multiples of 7 are :

252, 259 and 266

Note that $252 = 36 \times 7$, $259 = 37 \times 7$ and $266 = 38 \times 7$.

Check : Sum of the three multiples obtained $= 252 + 259 + 266 = 777$.

Hence, the solution is correct.

Remark : Sometimes it is more convenient to take the variable not as x, but a multiple of x. For example, it would have been better to assume above that the first number is $7x$ (the xth multiple of 7). Then the next two numbers would have been $7(x + 1)$ and $7(x + 2)$. The resulting equation would be

$$7x + 7(x+1) + 7(x + 2) = 777$$

or $$x + (x + 1) + (x + 2) = 111 \qquad \text{[Dividing both sides by 7]}$$

or $$3x + 3 = 111$$

or $$x = 36$$

Hence, the three consecutive multiples are

$$36 \times 7,\ 37 \times 7 \text{ and } 38 \times 7$$

i.e., 252, 259 and 266.

Example 7 : A fruit vendor buys some oranges at the rate of Rs 5 per orange. He also buys an equal number of bananas at the rate of Rs 2 per banana. He makes a 20% profit on oranges and a 15% profit on bananas. At the end of the day, all the fruit is sold out. His total profit is Rs 390. Find the number of oranges purchased.

Solution I : By reading the problem, we find that the following facts are *given* :

(1) Price of an orange is Rs 5.
(2) Price of a banana is Rs 2.
(3) Number of bananas = Number of oranges.
(4) Profit on oranges is 20 %.
(5) Profit on bananas is 15 %.
(6) All the fruit is sold out.
(7) Total profit = Rs 390.

We are *required* to find the number of oranges purchased.

II. Since we are required to find the number of oranges purchased, let us suppose that the number of oranges purchased = x.

III. Now we look at the statements given in the problem and listed in Step I above one by one and try to translate them into mathematical statements. According to (3), and Step II,

Number of oranges = Number of bananas = x

$$\text{Price of oranges} = \text{Rs } 5 \times x = \text{Rs } 5x \qquad \text{(A)}$$

$$\text{Price of bananas} = \text{Rs } 2 \times x = \text{Rs } 2x \qquad \text{(B)}$$

By (4),

$$\text{Profit on oranges} = \text{Rs } (5x \times \frac{20}{100}) \qquad \text{[Using (A)]}$$

$$= \text{Rs } x \qquad \text{(C)}$$

By (5),

$$\text{Profit on bananas} = \text{Rs}\left(2x \times \frac{15}{100}\right) \qquad \text{[Using (B)]}$$

$$= \text{Rs}\left(\frac{3}{10}x\right) \qquad \text{(D)}$$

By (7),

Rs 390 = Total Profit = Profit on oranges + Profit on bananas

$$= \text{Rs } x + \text{Rs}\left(\frac{3}{10}x\right) \qquad \text{[Using (C) and (D)]}$$

$$= \text{Rs}\left(x + \frac{3}{10}x\right)$$

$$= \text{Rs}\left(\frac{13}{10}x\right)$$

This gives $\quad 390 = \dfrac{13}{10}x \qquad$ (E)

Solving the above linear equation, we get

$$\frac{10}{13} \times 390 = \frac{10}{13} \times \frac{13}{10}x$$

or $\quad 300 = x$

Hence, the number of oranges purchased = 300

IV. Now we check the solution. We may begin by checking that $x = 300$ satisfies equation (E). But this would only tell us that we have solved equation (E) correctly. It does not tell us that 300 oranges were actually purchased. To check this, *we must verify the solution against the given problem*. Let us do that.

If 300 oranges are purchased at Rs 5 each and sold at a profit of 20 %, the profit on oranges would be

$$\text{Rs}\left(300\times5\times\frac{20}{100}\right),\ \text{i.e., Rs } 300 \qquad \text{(F)}$$

If 300 bananas (same number as oranges) are purchased at Rs 2 each and sold at a profit of 15 %, the total profit from bananas would be

$$\text{Rs}\left(300\times2\times\frac{15}{100}\right),\ \text{i.e., Rs } 90 \qquad \text{(G)}$$

From (F) and (G), the total profit of the vendor is

Rs 300 + Rs 90, i.e., Rs 390

This checks the solution.

Example 8 : The sides (other than hypotenuse) of a right triangle are in the ratio 3 : 4. A rectangle is described on its hypotenuse, the hypotenuse being the longer side of the rectangle (Fig. 8.1). The breadth of the rectangle is four-fifth of its length. Find the shortest side of the right triangle, if the perimeter of the rectangle is 180 cm.

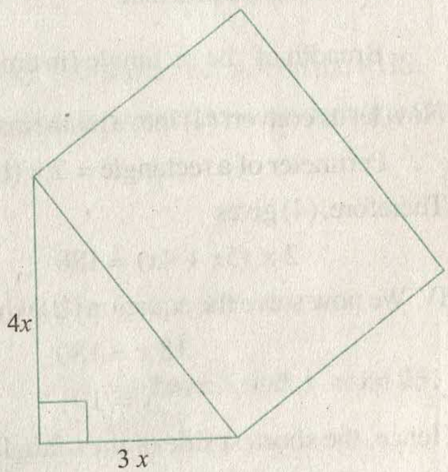


Fig. 8.1

Solution I : Reading the problem carefully, we find that the following facts are *given :*

(1) The sides of the right triangle are in the ratio 3 : 4.

(2) Length of the rectangle = Hypotenuse of the right triangle.

(3) Breadth of the rectangle = $\frac{4}{5}\times$ (Length of the rectangle).

(4) Perimeter of the rectangle = 180 cm.

We are *required* to find the shortest side of the triangle.

II. Since a side of the triangle is to be determined therefore, due to fact (1) above, it would be convenient to assume that the shortest side (in cm) is $3x$. Then the other side (in cm) of the triangle will be $4x$.

III. We now try to convert the information given in the problem [facts (1) to (4) above)] into mathematical statements. (1) has already been converted into $3x$ and $4x$ in Step II above. Consider (2) now. This tells us that the length of the rectangle is the same as the hypotenuse of the right triangle. Let us calculate the hypotenuse.

By Pythagoras Theorem,

$$\text{Hypotenuse (in cm)} = \sqrt{\text{Sum of squares of sides}}$$

$$= \sqrt{(3x)^2 + (4x)^2}$$

$$= \sqrt{25x^2}$$

$$= 5x$$

$\therefore$ Length of the rectangle (in cm) $= 5x$ (A)

From (3), we find that

Breadth of the rectangle (in cm) $= \frac{4}{5} \times 5x = 4x$ (B)

Now let us convert (4) into a mathematical statement. Recall that

Perimeter of a rectangle $= 2 \times (\text{Length} + \text{Breadth})$

Therefore, (4) gives

$$2 \times (5x + 4x) = 180 \qquad \text{(C)}$$

IV. We now solve the equation (C) which may be written as

$$18\,x = 180$$

$$\therefore \quad x = 10$$

Hence, the shortest side of the triangle (in cm) $= 3x = 3 \times 10 = 30$

Check : We now check the solution against the given problem. The shortest side of the triangle is 30 cm. Therefore, since the sides are in the ratio 3 : 4, the other side is 40 cm. This means that the

$$\text{Hypotenuse of the triangle} = \sqrt{(30^2 + 40^2)}\ \text{cm} = \sqrt{2500}\ \text{cm} = 50\ \text{cm}$$

Thus, the length of the rectangle = 50 cm and its breadth is $\frac{4}{5} \times 50$ cm $= 40$ cm

The perimeter of the rectangle is now $2 \times (50 + 40)$ cm, i.e., 180 cm.

Thus, the solution is correct.

Example 9 : The difference between two positive integers is 50. The ratio of these integers is 1 : 3. Find these integers.

Solution I : We are given two relations between two integers. We have to find the values of the two integers. Let us denote the smaller of the two integers to be found by the unknown x.

II. The first statement of the problem tells us that the difference between the two integers is 50. Therefore, the greater integer is $x + 50$. Thus, the two integers are x and $x + 50$.

III. We now read the next statement in the problem. It says that the ratio of the two integers is 1 : 3. Translating this relation into a mathematical statement, we get

$$\frac{1}{3} = \frac{x}{x+50} \qquad (1)$$

Since 1 is less than 3, we put the smaller integer x in the numerator on the RHS.

IV. No more information is given in the problem. So, we proceed to solve equation (1) above. By cross multiplication,

$$1 \times (x + 50) = 3 \times x$$

or $$x + 50 = 3x$$

or $$50 = 2x$$

or $$x = 25$$

V. The two integers were expressed as x and $x + 50$. We have found x to be 25. Hence, the two integers are 25 and 25 + 50, i.e., 25 and 75.
Thus, the required integers are 25 and 75.

VI. Let us now check the solution. The first statement in the problem tells us that the difference between the two integers is 50. We find that $75 - 25 = 50$. So far, the solution is correct. The second statement in the problem says that the ratio of the two integers is 1 : 3. Now 25 : 75 is the same as the ratio 1 : 3. Hence, the solution is correct.

Remark : There would have been no harm in checking at the end of Step V that $x = 25$ satisfies equation (1). But this check would not have been enough to tell us that we

have found the correct solution to the given problem. So we must check the solution against the given problem as in Step VI above.

Alternate Solution : Since the numbers are in the ratio 1 : 3, we may assume that the two numbers are x and $3x$. Since the difference between the two numbers is 50, we have

$3x - x = 50$ [Observe that $3x$ is the bigger number.]

or $2x = 50$

or $x = 25$

Hence, the two numbers are 25 and 75.

Example 10 : The sum of the digits of a two-digit number is 7. The number obtained by interchanging the digits exceeds the original number by 27. Find the number.

Solution : We are given that the required *number is a two-digit number*. Therefore, to find this number, we have to determine its units digit and its tens digit.

Suppose that the units digit is x. Since it is given that the sum of the digits of the number is 7, the tens digit must be $(7 - x)$. Thus, the number in the expanded notation is

$$(7 - x) \times 10 + x, \text{ i.e., } 70 - 9x \qquad (1)$$

[Recall that $25 = 2 \times 10 + 5, 81 = 8 \times 10 + 1, 36 = 3 \times 10 + 6$, etc. in the expanded notation.]

Let us now interchange the digits of the given number. Then the units digit becomes $(7 - x)$ and the tens digit becomes x. This new number, expressed in the expanded notation, is

$$x \times 10 + (7 - x), \text{ i.e., } 9x + 7 \qquad (2)$$

It is given that the new number exceeds the given number by 27. Therefore, from (1) and (2), we get

$(9x + 7) - (70 - 9x) = 27$

or $18x - 63 = 27$

or $18x = 90$ (Transposing -63)

or $x = 5$ (Dividing by 18)

Thus, units digit $= x = 5$,

tens digit $= 7 - x = 7 - 5 = 2$

Hence, the required number is 25.

Check : Interchanging the digits, we get 52. Now

$52 - 25 = 27$

This checks the solution.

Example 11 : A motorboat goes downstream in a river and covers the distance between

two coastal towns in five hours. It covers this distance upstream in six hours. If the speed of the stream is 2 km/h, find the speed of the boat in still water.

Solution : Since we have to find the speed of the boat in still water, let us suppose that it is x km/h. This means that while going downstream, the speed of the boat will be $(x+2)$ km/h because the current of water is pushing the boat at 2 km/h in addition to its own speed x km/h. While going upstream, the boat has to work against the current of water. Therefore, its speed upstream will be $(x-2)$ km/h.

It is given that while going downstream, the boat takes five hours to cover the distance between two coastal towns, say A and B. Now

Speed of the boat downstream $= (x+2)$ km/h

Distance covered in 1 hour $= (x+2)$ km

$\therefore$ Distance covered in 5 hours $= 5 \times (x+2)$ km

Hence, the distance between A and B is $5(x+2)$ km. (1)

Speed of the boat upstream $= (x-2)$ km/h

Distance covered in 1 hour $= (x-2)$ km

$\therefore$ Distance covered in 6 hours $= 6 \times (x-2)$ km

Hence, the distance between A and B is $6(x-2)$ km also. (2)

Since the distance between A and B is fixed, therefore, comparing (1) and (2), we get

$$5(x+2) = 6(x-2)$$

Solving this linear equation, we get $x = 22$. Thus, the required speed of the boat is 22 km/h.

EXERCISE 8.2

1. The difference between two positive integers is 36. The quotient, when one integer is divided by the other is 4. Find the two integers.
2. The sum of two positive integers is 98. The integers are in the ratio 3 : 4. Find the integers.
3. The denominator of a rational number is greater than its numerator by 8. If the numerator is increased by 17 and the denominator is decreased by 1, the number obtained is $\frac{3}{2}$. Find the rational number.
4. One number is 3 times another number. If 15 is added to both the numbers, then one of the new numbers becomes twice that of the other new number. Find the numbers.

5. The sum of two consecutive multiples of 5 is 55. Find these two multiples.

6. The sum of three consecutive multiples of 6 is 666. Find these multiples.

7. The sum of three consecutive multiples of 9 is 999. Find these multiples.

8. The ages of Ruby and Reshma are in the ratio 5 : 7. Four years later, their ages will be in the ratio 3 : 4. Find their ages.

9. Five years ago, Luckee was three times as old as Lovely. 10 years later, Luckee would be twice as old as Lovely. How old are they now?
[*Hint :* First find how old they were five years ago.]

10. The perimeter of a rectangle is 240 cm. If its length is decreased by 10 % and its breadth is increased by 20 %, we get the same perimeter. Find the length and the breadth of the rectangle.
[*Hint :* If l and b denote respectively the length and the breadth, then $b = 120 - l$)]

11. Sum of the digits of a two-digit number is 9. The number obtained by interchanging the digits exceeds the given number by 27. Find the given number.

12. Sum of the digits of a two-digit number is 12. The given number exceeds the number obtained by interchanging the digits by 36. Find the given number.

13. Each side of a triangle is increased by 10 cm. If the ratio of the perimeters of the new triangle and the given triangle is 5 : 4, find the perimeter of the given triangle.
[*Hint :* If the sides are a, b, c then the perimeter is $a + b + c = x$ say.]

14. Kanchan receives a certain amount of money on her retirement from her employer. She gives half of this money and an additional sum of Rs 10000 to her daughter. She also gives one-third of the money received and an additional sum of Rs 3000 to her son. If the daughter gets twice as much as the son, find the amount of money Kanchan received on her retirement.

15. A motorboat covers a certain distance downstream in a river in five hours. It covers the same distance upstream in five hours and a half. The speed of water is 1.5 km/h. Find the speed of the boat in still water.

16. A steamer, going downstream in a river, covers the distance between two towns in 20 hours. Coming back upstream, it covers this distance in 25 hours. The speed of water is 4 km/h. Find the distance between the two towns.
[*Hint :* First find the speed of the steamer in still water.]

17. A race-boat covers a distance of 66 km downstream in 110 minutes. It covers this distance upstream in 120 minutes. The speed of the boat in still water is 34.5 km/h. Find the speed of the stream.

Things to Remember

1. In order to solve equations of the type $\frac{ax+b}{cx+d} = k, cx + d \neq 0$ where a, b, c, d and k are numbers, we write them as $ax + b = k(cx + d)$. This is called method of cross multiplication.
2. To solve a word problem, denote the unknown by some variable and translate the statements given in the problem step by step into a mathematical statement. Form relevant equalities and solve the corresponding equations.

---- As History Tells Us ----

You are already familiar with contributions made by the Indian mathematicians like *Aryabhata*, *Brahmagupta* and *Bhaskara* to the field of Algebra. However, the roots of the subject that we call Algebra today may be traced to antiquity, which we may term as the first round. Here, we find number puzzles that can be solved with an algebraic flavour. *Ahmes* (1550 B.C.) was a scribe who put down the mathematical knowledge of that time on a *papyrus*. A problem given in this papyrus is: *Mass, its whole, its seventh, it makes 19*. ($x + \frac{x}{7}$) = 19 in our terminology).

In the second round, algebra appears in a geometric form in Greek quarters. Now Greeks were proficient in Geometry but somehow, they had no algebra at all. They carried out modern algebraic operations with devious geometric processes. For example, the Book II of *Euclid* contains geometric version of several algebraic identities. All these are proved by dissecting geometric figures into suitable pieces. For example, the identity $(a + b)^2 = a^2 + 2ab + b^2$, proved by early Pythagoreans is stated by Euclid is as follows :

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the two parts.

In the third round, algebra appears in fanciful oriental problems relating to human affairs (of which some examples were given in Class VII) as also in real

life problems. Consider the following problems, the first from a Greek Anthology and the second from China.

I. *Statue A* : How heavy is the base on which I stand together with myself?
Statue B : My base together with myself weighs the same number of talents.
Statue A : I alone weigh twice as much as your base.
Statue B : I alone weigh three times as much as your base.

(You may easily find the solution by assuming the total weight to be w and the weight of the base of the Statue B to be x.)

II. If a cock is worth 5 *sapeks*; a hen 3 *sapeks*; and 3 chickens together, 1 *sapek*, how many cocks, hens, and chickens, 100 in all, will together be worth 100 *sapeks*?

(You may find the solution by trial and error ; remember *sapek* is a unit of money).

In your higher classes, you will learn about several other types of Algebra.

CHAPTER

9

PARALLEL LINES

9.1 Introduction

You have learnt about parallel lines, transversals and some of their properties in earlier classes. You may recall the following facts :

1. Lines in a plane which do not intersect are called *parallel lines*.
2. A line which intersects two or more given lines in *distinct* points is called a *transversal* to the given lines.
3. The perpendicular distance between two parallel lines is the same everywhere and is called the distance between the two parallel lines.
4. If two parallel lines are intersected by a transversal, then
   (i) Each pair of corresponding angles are equal.
   (ii) Each pair of alternate interior angles are equal.
   (iii) Interior angles on the same side of the transversal are supplementary.
5. If two lines are intersected by a transversal, then they are parallel if any one of the following is true :
   (i) Any pair of corresponding angles are equal.
   (ii) Any pair of alternate interior angles are equal.
   (iii) Any pair of interior angles on the same side of the transversal are supplementary.

In this Chapter, we shall learn some more properties relating to parallel lines and apply this knowledge in (i) dividing a line segment into a given number of equal segments (parts) and (ii) dividing a line segment in a given ratio.

9.2 Lines Parallel to the Same Line

Suppose, we have a fixed line l in a plane. Let us consider two lines m and n in the plane, each parallel to l (Fig. 9.1). Thus, $m \parallel l$ and $n \parallel l$. What can you say about these two lines m and n? Are these two lines parallel or not? Let us examine.

Fig. 9.1

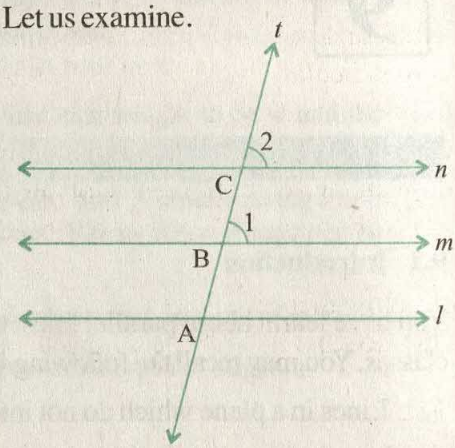


Fig. 9.2

Activity 1 : Draw a line l on a sheet of paper. Using ruler and set-square, draw two lines m and n, each parallel to l. Now draw a transversal t, intersecting l, m and n at A, B and C respectively, forming corresponding angles 1 and 2 at B and C (Fig. 9.2).

Measure $\angle 1$ and $\angle 2$ with the help of a protractor and find the difference $\angle 1 - \angle 2$. Repeat the above activity by drawing two other figures involving the same type of parallel lines. For convenience, in each case, label the figure in the same way. Number the figures as 1, 2 and 3. Find $\angle 1$, $\angle 2$ and difference $\angle 1 - \angle 2$ for each case and record your observations in the form of a table as shown below :

Figure	$\angle 1$	$\angle 2$	$\angle 1 - \angle 2$
1.			
2.			
3.			

What do you observe? In each case, we observe that the difference $\angle 1 - \angle 2$ is nearly zero. The non-zero difference may be due to inaccurate measurements. Thus, in each case, it appears that $\angle 1 = \angle 2$.

As $\angle 1$ and $\angle 2$ are corresponding angles, it follows that in each case $m \parallel n$.

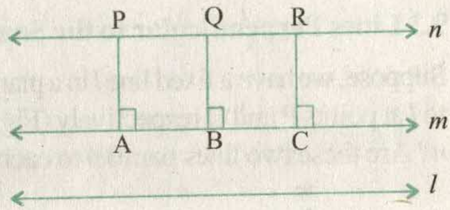


Fig. 9.3

Activity 2 : As in the case of Activity 1, draw any line l on a sheet of paper and draw two lines m and n, each parallel to l, using ruler and set-square. Now take three points P, Q and R on line n and from these points, draw perpendiculars PA, QB and RC, respectively to line m such that A, B and C lie on m (Fig. 9.3). Repeat the above activity by drawing two other figures involving the same type of parallel lines and perpendiculars. For convenience, in each case, label the figure in the same way and number the figures as 1, 2 and 3.

Now measure PA, QB and RC in each case and find the differences PA – QB, QB – RC and RC – PA. Record your observations in the form of a table as shown below :

Figure	PA	QB	RC	PA – QB	QB – RC	RC – PA
1.						
2.						
3.						

What do you observe? You will observe that, in each case, differences of perpendicular distances PA – QB, QB – RC and RC – PA are nearly zero. Here, again the non-zero difference may be due to inaccurate measurements.

Thus, in each case, it appears that PA = QB = RC. In other words, perpendicular distances between the two lines m and n are equal everywhere. Thus, $m \parallel n$.

The above two activities illustrate the following proposition :

Two lines which are parallel to the same line are parallel to each other.

In this case, we also say that the three lines are parallel to each other.

Remark : We may restate the above property as follows :

Two lines which are parallel to the same line cannot intersect. In other words, *two intersecting lines cannot both be parallel to the same line*.

9.3 Lines Perpendicular to the Same Line

Suppose, we have a fixed line l in a plane. Let us consider two lines m and n perpendicular to l at points P and Q respectively (Fig. 9.4). What can you say about the two lines m and n? Are these two lines parallel to each other or not? Let us examine.

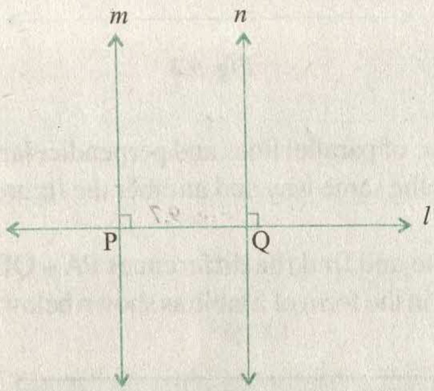


Fig. 9.4

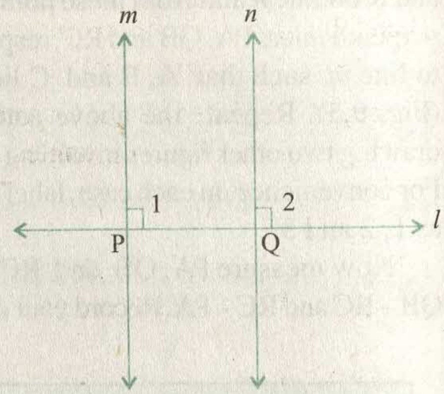


Fig. 9.5

Activity 3 : Draw any line l on a sheet of paper and take any two points P and Q on it. Using ruler and set-square, draw two lines perpendicular to l forming corresponding angles 1 and 2 at P and Q, respectively (Fig. 9.5). We note that $\angle 1 = 90° = \angle 2$. We also note that $\angle 1$ and $\angle 2$ are corresponding angles. As the above corresponding angles are equal, we have $m \parallel n$.

This activity illustrates the following proposition :

Two lines which are perpendicular to the same line are parallel to each other.

We can also verify the above property by taking three points A, B and C on m, drawing perpendiculars AM, BN and CR respectively to n and then measuring AM, BN and CR as done in Activity 2 above. We will find that AM = BN = CR and, therefore, $m \parallel n$.

Let us take some examples to illustrate these properties.

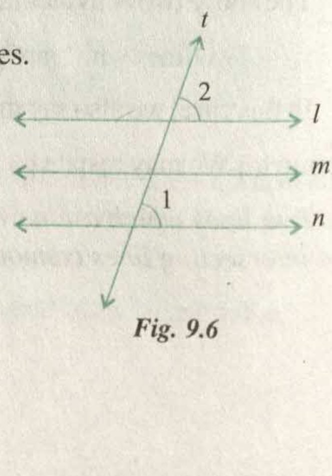


Fig. 9.6

Example 1 : In Fig 9.6, $l \parallel m$ and $m \parallel n$. If $\angle 1 = 70°$, find $\angle 2$.

Solution: $l \parallel m$ and $m \parallel n$ (Given)

Therefore, $l \parallel n$ (Lines parallel to the same line)

Hence, $\angle 2 = \angle 1 = 70°$

(Corresponding angles)

Example 2 : In Fig 9.7, $m \perp l$, $n \perp l$ and transversal t forms $\angle 1$ and $\angle 2$ with n and m respectively. If $\angle 1 = 80°$, find $\angle 2$.

Solution : $m \perp l$ and $n \perp l$ (Given)

Therefore, $m \parallel n$ (Lines perpendicular to the same line)

Hence, $\angle 2 = \angle 1 = 80°$ (Alternate interior angles)

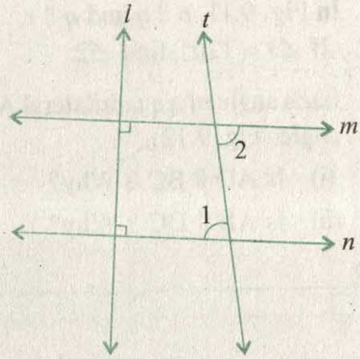


Fig. 9.7

EXERCISE 9.1

1. In Fig. 9.8, AB || DC and EF || AB.
   (i) Is EF || DC also? Why?
   (ii) How many pairs of parallel line segments are there in the figure? Name each of them.

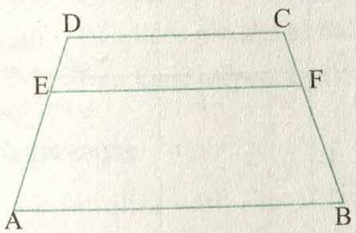


Fig. 9.8

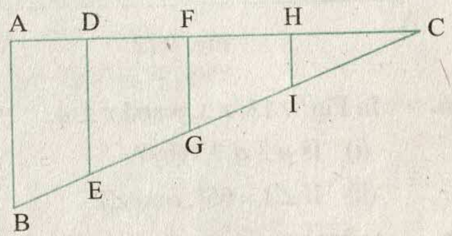


Fig. 9.9

2. In Fig. 9.9, line segments DE, FG and HI are each parallel to side AB of Δ ABC. How many pairs of parallel line segments are there in the figure? Name each of them.

3. In Fig. 9.10, $l \parallel m$ and $l \parallel n$.
   (i) Is $m \parallel n$? Why?
   (ii) Find the value of x.

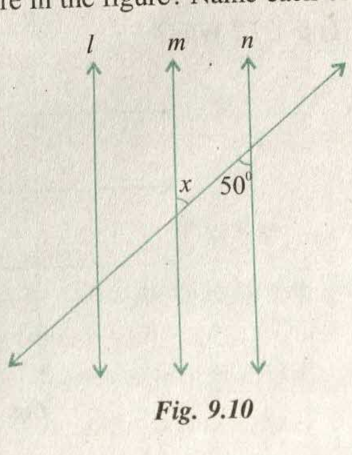


Fig. 9.10

4. In Fig. 9.11, $p \parallel q$ and $q \parallel r$.
   If $\angle 1 = 120°$, find $\angle 2$.

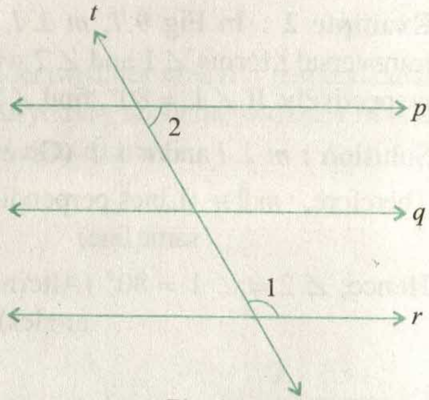


Fig. 9.11

5. Each angle of a quadrilateral ABCD is a right angle (Fig. 9.12).
   (i) Is AD $\parallel$ BC ? Why?
   (ii) Is AB $\parallel$ DC ? Why?

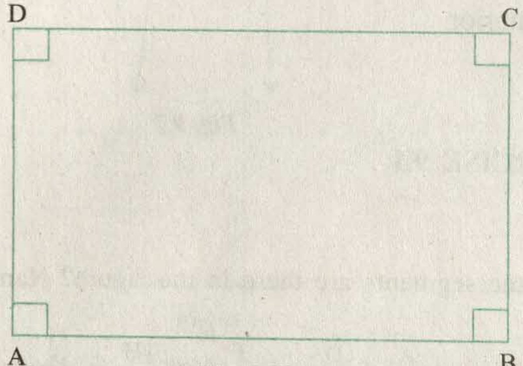


Fig. 9.12

6. In Fig. 9.13, $r \perp p$ and $r \perp q$.
   (i) Is $p \parallel q$? Why?
   (ii) If $\angle 1 = 65°$, find $\angle 2$.

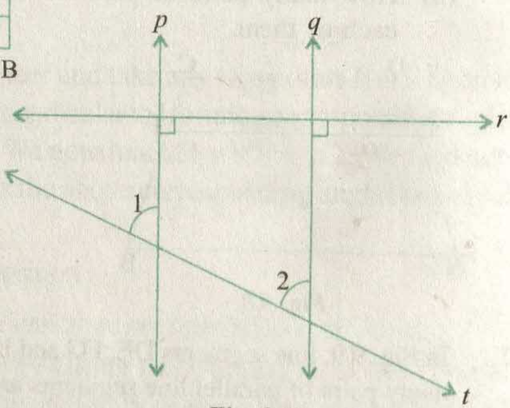


Fig. 9.13

7. In Fig. 9.14, $l \parallel m$, $p \perp m$ and $p \perp n$.
   (i) Is $m \parallel n$? Why?
   (ii) Is $l \parallel n$? Why?
   (iii) Is $p \perp l$? Why?

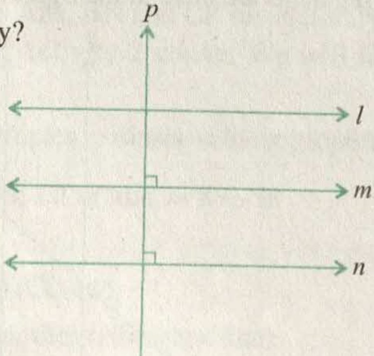


Fig. 9.14

8. In Fig. 9.15, AB, EF and DC are each perpendicular to BC. How many pairs of parallel line segments are there in the figure? Name each of them.

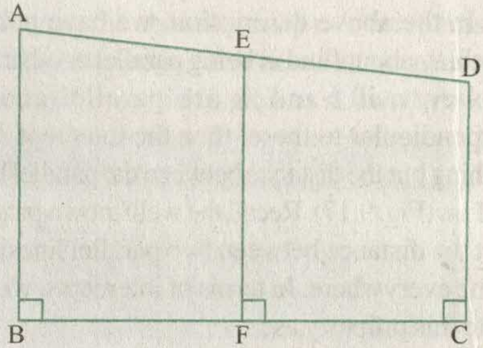


Fig. 9.15

9. Write true (T) or false (F) for the following statements :
   (i) Two lines parallel to the same line are parallel to each other.
   (ii) Two lines parallel to the same line are perpendicular to each other.
   (iii) Two lines perpendicular to the same line are parallel to each other.
   (iv) Two lines perpendicular to the same line are perpendicular to each other.
   (v) Two lines parallel to the same line intersect each other.
   (vi) Two lines perpendicular to the same line intersect each other.

9.4 Intercepts

You are familiar with Fig. 9.16 from your earlier classes. There are two lines l and m and a third line t intersects them in two distinct points A and B respectively. What is this third line called? It is called a *transversal.*

The above figure can also be described in a slightly different way. We say that two lines l and m intersect a third line t in distinct points A and B. Thus, the two lines l and m cut off line segment AB on the line t. We give this line segment a special name – *intercept*. Thus :

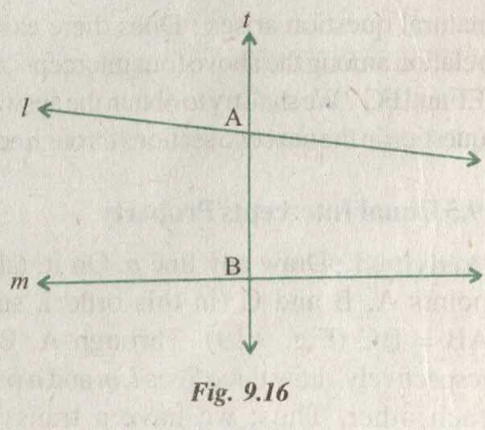


Fig. 9.16

If a transversal t intersects two lines l and m in distinct points A and B, then the lines l and m are said to make an intercept AB on t.

Observe that intercept AB is a line segment. However, the term 'intercept' is also used for the length of the intercept AB.

In the above description, we have not said anything about l and m being parallel or otherwise. However, if l and m are parallel and t is perpendicular to these, then the intercept AB is nothing but the distance between the parallel lines l and m. (Fig. 9.17). Recall the well known property, that the distance between two parallel lines is the same everywhere. In terms of intercepts, we may state this property as :

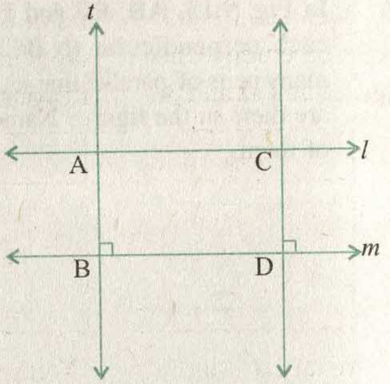


Fig. 9.17

Two parallel lines make equal intercepts on all transversals perpendicular to them.

Now if l, m and n are three parallel lines (Fig. 9.18) and a transversal p intersects the same in three points A, B and C, respectively, then we say that the pairs of lines l, m and m, n make intercepts AB and BC, respectively on p.

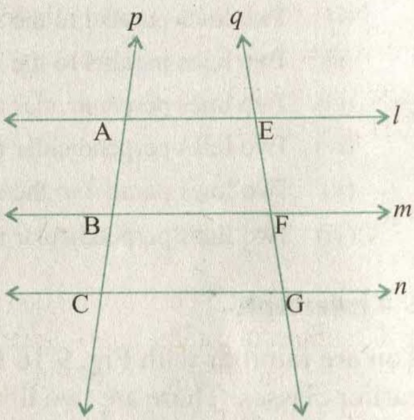


Fig. 9.18

Similarly, if a tansversal q intersects l, m and n in three points, E, F and G, respectively, then we say that the pairs of lines l, m and m, n make intercepts EF and FG, respectively on q. Now a natural question arises : Does there exist some relation among the above four intercepts AB, BC, EF and FG ? We shall try to obtain the answer to this question in the next two sections through activities :

9.5 Equal Intercepts Property

Activity 4 : Draw any line p. On it, take three points A, B and C (in this order), such that AB = BC (Fig. 9.19). Through A, B and C respectively, draw three lines l, m and n parallel to each other. Thus, we have a transversal p intersecting three parallel lines l, m and n such that AB = BC. Now draw another transversal q intersecting lines, l, m and n at E, F and G respectively.

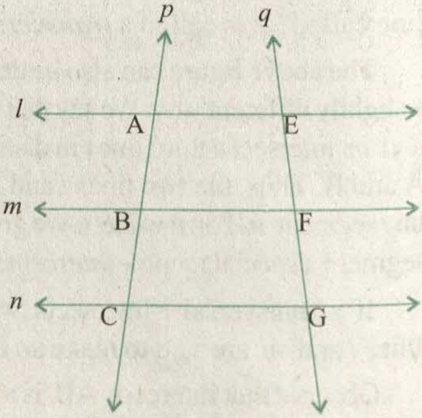


Fig. 9.19

Measure EF and FG and find EF – FG. Repeat the above activity two more times. For convenience, use the same letters to label the figure in each case and number the figures as 1, 2 and 3. Record your observations in the form of a table as shown below :

Figure	EF	FG	EF – FG
1.			
2.			
3.			

What do you observe? You will observe that in each case, EF – FG is nearly zero. Thus, in all cases, it appears that EF = FG.

Note that we have taken three parallel lines l, m and n such that they make equal intercepts AB and BC on transversal p. Then we obtained that the same three parallel lines make equal intercepts EF and FG on any other transversal q. This activity illustrates the following proposition :

If three parallel lines make equal intercepts on one transversal, then they make equal intercepts on any other transversal as well.

The above property holds good even if the number of parallel lines is more than three and accordingly, the number of intercepts more than two. For example, in Fig. 9.20, four parallel lines l_1, l_2, l_3 and l_4 make three intercepts AB, BC and CD on p and three intercepts, EF, FG and GH on q such that AB = BC = CD.

Since AB = BC, therefore, EF = FG.

Also, since BC = CD, therefore, FG = GH.

Hence, if AB = BC = CD, then EF = FG = GH.

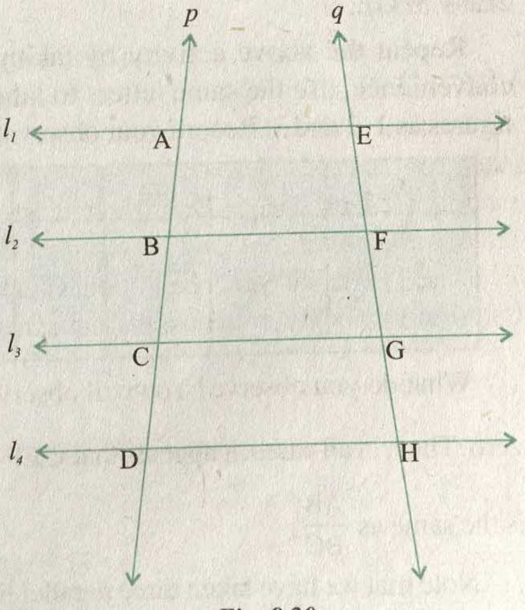


Fig. 9.20

Thus, in general, we may say :

If three or more parallel lines make equal intercepts on one transversal, then they make equal intercepts on any other transversal as well.

This property is referred to as the *Equal Intercepts Property*.

9.6 Proportional Intercepts Property

Activity 5 : Draw any line p and take three points A, B and C on it such that AB = x cm and BC = y cm, where x and y are certain numbers, say $x = 5$ and $y = 2$ (Fig.9.21). Through A, B and C respectively, draw lines l, m and n parallel to each other. Thus, we have a transversal p intersecting three parallel lines l, m and n such that $\frac{AB}{BC} = \frac{x}{y}\left(= \frac{5}{2} \text{ in this case}\right)$.

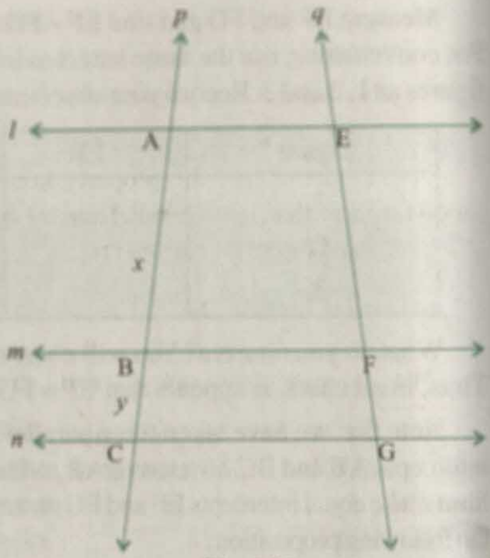


Fig. 9.21

Now draw another transversal q intersecting lines l, m and n at E, F and G respectively. Measure EF and FG and then find the value of $y.\text{EF} - x.\text{FG}$ (in this case $2\text{EF} - 5\text{FG}$).

Repeat the above activity by taking two other values of x and y each. For convenience, use the same letters to label the figure in each case and number the figures as 1, 2 and 3. Record your observations in the form of a table as given below :

Figure	EF	FG	x	y	y.EF	x.FG	y.EF − x.FG
1.							
2.							
3.							

What do you observe? You will observe that, in each case, $y.\text{EF} - x.\text{FG}$ is nearly zero. Thus, in all cases, it appears that $y.\text{EF} - x.\text{FG} = 0$. In other words, $\frac{\text{EF}}{\text{FG}} = \frac{x}{y}$, which is the same as $\frac{\text{AB}}{\text{BC}}$.

Note that we have taken three parallel lines l, m and n that make intercepts AB and BC on a transversal p in the ratio $\frac{x}{y}$ (in the present case $\frac{5}{2}$). Then we found that the same three parallel lines make intercepts EF and FG on any other transversal q in the same ratio $\frac{x}{y}$. This activity illustrates the following proposition :

If three parallel lines make intercepts on one transversal in a certain ratio, then they make intercepts in the same ratio on any other transversal.

In other words, three parallel lines intersecting any two transversals make intercepts on them in the same ratio.

As a matter of fact, this property, known as *Proportional Intercepts Property* holds good for more than three parallel lines also. Thus, we may say :

Proportional Intercepts Property :

Three or more parallel lines intersecting any two transversals make intercepts on them in the same proportion.

We now take some examples to illustrate these properties.

Example 3: A and B are points not lying on a line l (Fig. 9.22). C is the mid-point of AB, $AD \perp l$, $BF \perp l$ and $CE \perp l$.

(i) Is $AD \parallel CE$? Why?

(ii) Is $AD \parallel BF$? Why?

(iii) Is $AD \parallel CE \parallel BF$? Why?

(iv) Is E the mid-point of DF? Why?

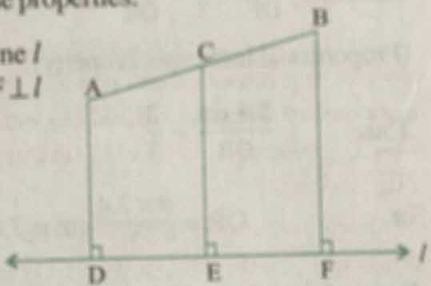


Fig. 9.22

Solution : (i) $AD \perp l$ and $CE \perp l$ (Given)

Therefore, $AD \parallel CE$. (Lines perpendicular to the same line)

(ii) $AD \perp l$ and $BF \perp l$ (Given)

Therefore, $AD \parallel BF$ (Lines perpendicular to the same line)

(iii) As $AD \parallel CE$ and $AD \parallel BF$, therefore $AD \parallel CE \parallel BF$ (Lines parallel to the same line)

(iv) For three parallel lines AD, CE and BF, AB is a transversal such that

$AC = CB$ (C is the mid-point)

Therefore, $DE = EF$ (Equal Intercepts Property)

i.e., E is the mid-point of DF.

Example 4 : In Fig. 9.23, PAQ is a line parallel to side BC of ΔABC. D is the mid-point of side AB and $DE \parallel BC$. Is E the mid-point of side AC?

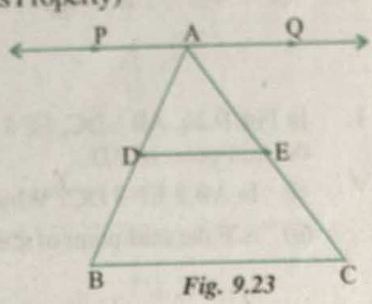


Fig. 9.23

Solution : $PAQ \parallel BC$ and $DE \parallel BC$ (Given)

Therefore, PAQ, DE and BC are three lines parallel to each other (Lines parallel to the same line)

AD = DB (D is mid-point of AB)

Therefore, AE = EC (Equal Intercepts Property)

i.e., E is the mid-point of side AC.

Example 5 : In Fig. 9.24, AB || CD || EF and BP || DQ || FR. If AC = 4 cm, CE = 6 cm and PQ = 2.4 cm, find QR.

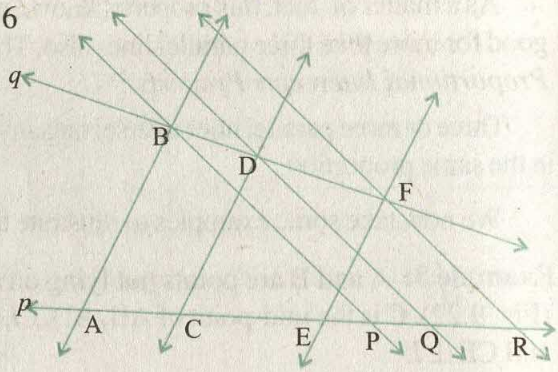


Fig. 9.24

Solution : $\frac{AC}{CE} = \frac{4}{6} = \frac{2}{3}$ (Given)

Therefore, $\frac{BD}{DF} = \frac{2}{3} = \frac{PQ}{QR}$

(Proportional Intercepts Property)

Thus, $\frac{2.4 \text{ cm}}{QR} = \frac{2}{3}$

or $QR = \frac{3 \times 2.4}{2}$ cm = 3.6 cm

Example 6 : In Fig. 9.25, PAQ || DE || BC. If AD = 3 cm, DB = 6 cm and EC = 8 cm, find AE.

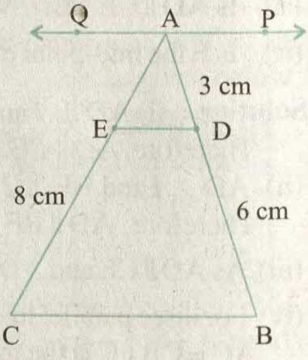


Fig. 9.25

Solution : (i) $\frac{AE}{EC} = \frac{AD}{DB}$ (Proportional Intercepts Property)

i.e., $\frac{AE}{8} = \frac{3}{6}$ cm

or $AE = \frac{8 \times 3}{6}$ cm = 4 cm

EXERCISE 9.2

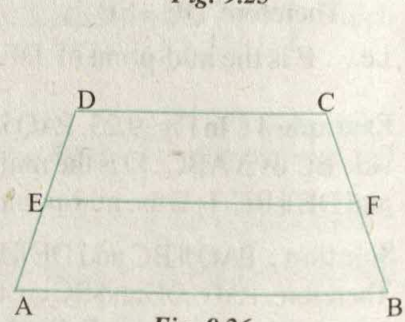


Fig. 9.26

1. In Fig. 9.26, AB || DC, EF || AB and E is the mid-point of AD.
   (i) Is AB || EF || DC? Why?
   (ii) Is F the mid-point of CB? Why?

2. ABC is an isosceles triangle with AB = AC (Fig. 9.27). Further, $l \parallel DE \parallel BC$ and D is the mid-point of AB.

(i) Is E the mid-point of AC? Why?

(ii) Is ΔADE also isosceles? Give reasons.

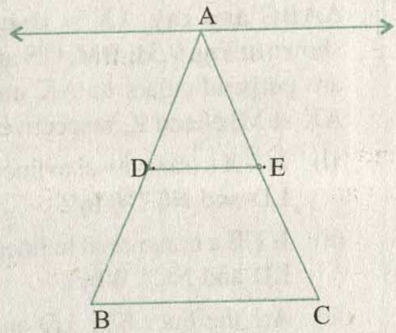


Fig. 9.27

3. In Fig. 9.28, $l \parallel DE \parallel BC$, $m \parallel EF \parallel AB$ and D is the mid-point of AB.

(i) Is E the mid-point of AC? Why?

(ii) Is F the mid-point of BC? Why?

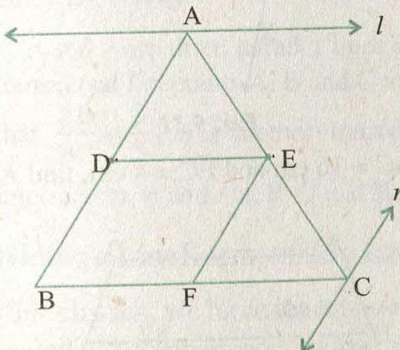


Fig. 9.28

4. In Fig. 9.29, $AP \parallel BQ \parallel CR$, $PX \parallel QY \parallel RZ$ and AB = BC. Is

(i) PQ = QR ? Why?

(ii) XY = YZ ? Why?

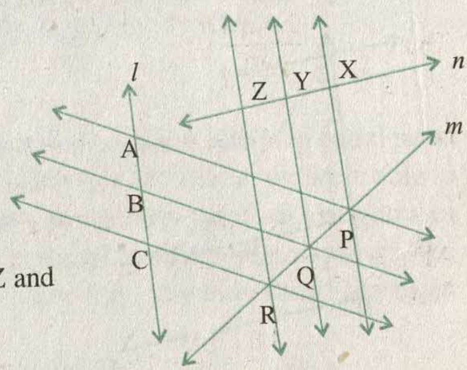


Fig. 9.29

5. In Fig. 9.30, AD is a median of the ΔABC, E is the mid-point of AD and $l \parallel DG \parallel BF \parallel m$.

(i) Is FG = GC ? Why?

(ii) Is AF = FG ? Why?

(iii) If AC = 4.5 cm, find AF.

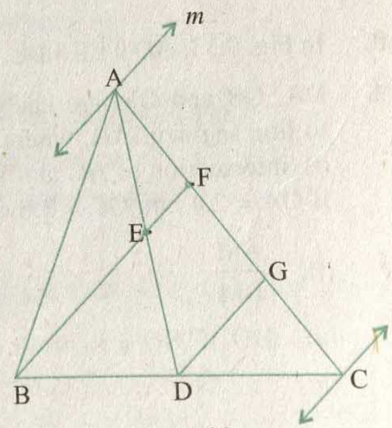


Fig. 9.30

6. D is the mid-point of side BC of ΔABC and ray AX is drawn as shown in Fig. 9.31. BM, CN and DL are perpendiculars to AX meeting AX at M, N and L, respectively.

   (i) Is AX a transversal to lines BM, LD and NC? Why?

   (ii) Is CB a transversal to lines BM, LD and NC? Why?

   (iii) Are the lines BM, LD and NC parallel to each other? Why?

   (iv) Is ML = LN? Why?

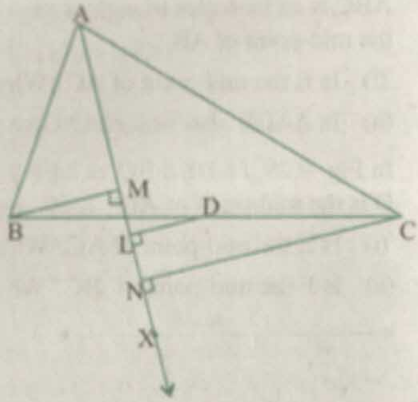


Fig. 9.31

7. In Fig. 9.32, $l \parallel ED \parallel CB$. If AB = 12 cm, AC = 16 cm and EC = 4 cm, find AD.

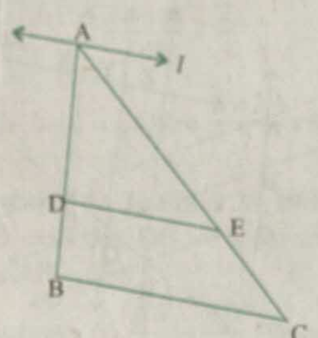


Fig. 9.32

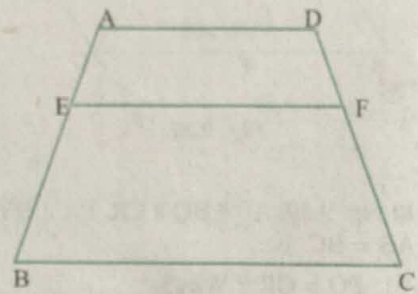


Fig. 9.33

8. In Fig. 9.33, $AD \parallel EF \parallel BC$. If EB = 2AE and DF = 1.5 cm, find the length of FC.

9. DA, CB and OM are each perpendicular to line segment AB, where O is the point of intersection of AC and DB (Fig. 9.34). If OA = 2.4 cm, OC = 3.6 cm, find :

   (i) $\dfrac{AM}{BM}$

   (ii) DO, if BO = 3 cm

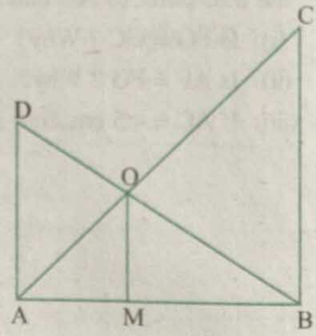


Fig. 9.34

10. A plot of land ABCD is subdivided into three plots AQPD, PQSR and RSBC as shown in Fig. 9.35. If CD = 30 m and DA ∥ PQ ∥ RS ∥ CB, find the lengths DP, PR and RC.

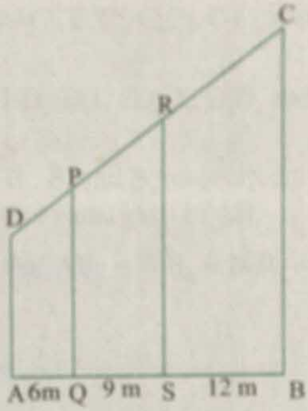


Fig. 9.35

11. l, m and n are three parallel lines intersected by a transversal p in points A, B and C respectively such that AB ≠ BC. q is another transversal which intersects l, m and n at D, E and F respectively. Is DE = EF ? Why ?

12. p, q and r are three parallel lines intersected by a transversal l in points A, B and C respectively such that $\frac{AB}{BC} = \frac{3}{5}$ · m is another transversal which intersects p, q and r at P, Q and R respectively. Is $\frac{PQ}{QR} = \frac{2}{5}$? Why?

9.7 Dividing a Line Segment into Equal Parts

In earlier classes, we have learnt how to divide a line segment into two equal parts using ruler and compasses. You may recall that using this technique, we were able to divide a line segment into, say, 4, 8, 16, etc. equal parts. Now we shall learn how to divide a line segment into any number of equal parts using ruler and compasses. We shall illustrate the process through an example. For this, you may recall the basic constructions learnt in Class VI.

Example 7 : Divide a given line segment AB of length 8 cm into five equal parts.

Solution : We go through the following steps for this construction :

1. Draw AB = 8 cm.
2. Draw a ray AC such that AC is not in the same line as AB (Fig. 9.36).
3. On ray AC, starting from point A, mark *five* equal line segments AC_1, C_1C_2, C_2C_3, C_3C_4 and C_4C_5 of any measurement with the help of compasses.
4. Join C_5B.
5. Through the points C_1, C_2, C_3 and C_4, draw lines parallel to C_5B intersecting AB at the points B_1, B_2, B_3 and B_4 respectively.

   Line segments AB_1, B_1B_2, B_2B_3, B_3B_4 and B_4B are the required five equal parts of AB.

Remark : You can easily see that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B$ by Equal Intercepts Property.

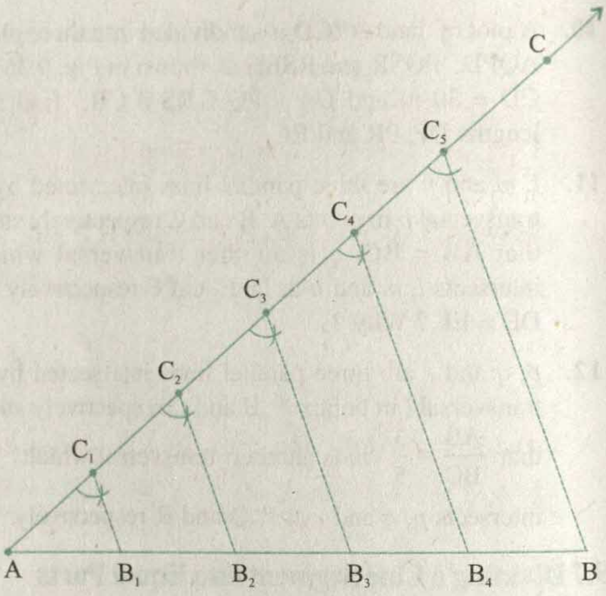


Fig. 9.36

Alternate Method :

1. Draw AB = 8 cm.
2. Draw a ray AC such that AC is not in the same line as AB.
3. Draw a ray BD parallel to CA as shown in Fig. 9.37.

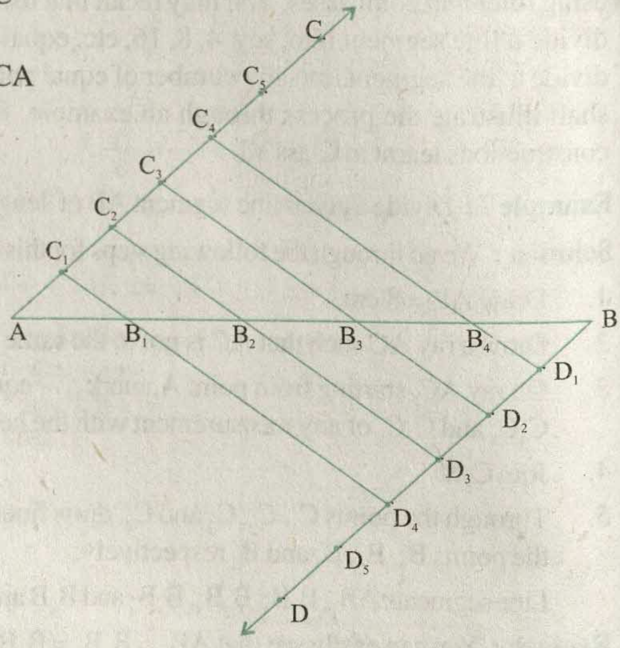


Fig. 9.37

4. On AC, starting from A, mark *five* equal line segments AC_1, C_1C_2, C_2C_3, C_3C_4 and C_4C_5 of any measurement.
5. On BD, starting from B, mark *five* equal line segments BD_1, D_1D_2, D_2D_3, D_3D_4 and D_4D_5 of the same length as in Step 4.
6. Join C_1D_4, C_2D_3, C_3D_2 and C_4D_1 to intersect AB at B_1, B_2, B_3 and B_4 respectively. Then, AB_1, B_1B_2, B_2B_3, B_3B_4 and B_4B are the required five equal parts of AB.

Remark : By joining AD_5 and C_5B, you can easily see that $AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B$.

9.8 Dividing a Line Segment Internally in a Given Ratio

Again we shall explain this construction through an example.

Example 8 : Divide a line segment AB of length 5 cm say, in the ratio 3 : 4 internally.

Solution : We go through the followings steps for this construction :

1. Draw a line segment AB of length of 5 cm.
2. Draw a ray AC such that AC is not in the same line as AB (Fig. 9.38).
3. On ray AC, mark *seven* (3 + 4) equal line segments AC_1, C_1C_2, C_2C_3, C_3C_4, C_4C_5, C_5C_6 and C_6C_7, using compasses.
4. Join C_7B.
5. Starting from A and counting *three* line segments just drawn, we reach C_3. Through this point C_3, draw a line parallel to C_7B intersecting AB at a point P.

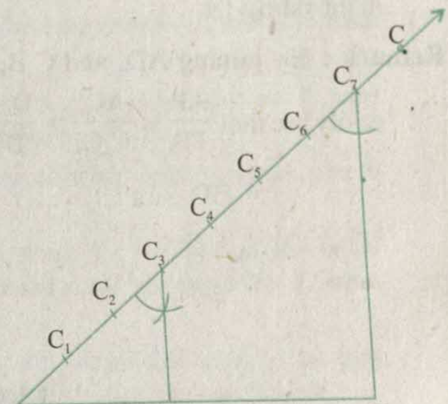


Fig. 9.38

P divides internally the line segment AB into two parts AP and PB, which are in the ratio of 3 : 4. We can check it by actually measuring the line segments AP and BP.

Remark : The fact that $\frac{AP}{PB} = \frac{AC_3}{C_3C_7} = \frac{3}{4}$ follows easily from the Proportional Intercept Property.

Alternate Method :

1. Draw a line segment AB of length 5 cm.

2. Draw a ray AC such that AC is not in the same line as AB.
3. Draw a ray BD parallel to CA as shown in Fig. 9.39.
4. On AC, starting from A, mark *seven* (3 + 4) equal line segments AC_1, C_1C_2, C_2C_3, C_3C_4, C_4C_5, C_5C_6 and C_6C_7.
5. On BD, starting from B, mark *seven* (3 + 4) equal line segments BD_1, D_1D_2, D_2D_3, D_3D_4, D_4D_5, D_5D_6 and D_6D_7 of the same length as in Step 4.
6. Counting *three* line segments along AC, we get the point C_3. Counting *four* line segments along BD, we get D_4. Join these points C_3 and D_4 intersecting AB at P. Then P is the required point which divides AB internally in the ratio 3 : 4.

Remark : By joining AD_7 and C_7B, you can easily see that $\frac{AP}{PB} = \frac{AC_3}{C_3C_7} = \frac{D_7D_4}{D_4B} = \frac{3}{4}$.

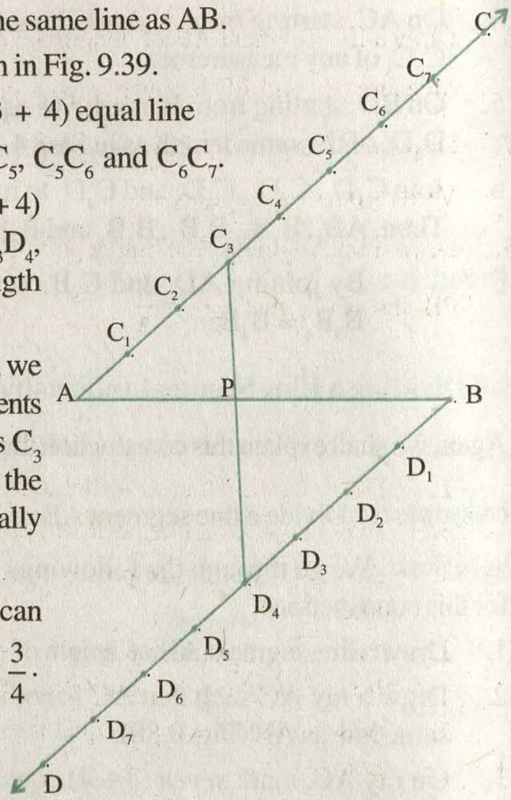


Fig. 9.39

EXERCISE 9.3

1. Draw a line segment AB of length 7.5 cm and divide it into three equal parts. Measure the length of each part.
2. Draw a line segment AB of length 8.5 cm and divide it into five equal parts. Measure the length of each part.
3. Draw a line segment PQ of length 6.6 cm. Divide it into six equal parts.
4. Draw a line segment PQ of length 12 cm. Divide it internally in the ratio 3 : 5. Verify your construction by measurement.
5. Draw a line segment MN of length 10 cm. Divide it internally in the ratio 2 : 3. Measure the smaller part.

6. Draw a line segment AB of length 5.6 cm. Locate a point P on AB such that $\frac{AP}{PB} = \frac{2}{5}$. Measure AP and PB and verify your construction.

7. Draw a line segment MN of length 6 cm. Locate a point P on MN such that $\frac{MP}{PN} = \frac{1}{2}$. Verify your construction.

8. Draw a line segment AB. Find a point P on AB such that AP : PB = 2 : 3. Measure AP, PB and verify that $\frac{AP}{PB} = \frac{2}{3}$.

Things to Remember

1. Two lines which are parallel to the same line are parallel to each other.
2. Two intersecting lines cannot both be parallel to the same line.
3. Two lines which are perpendicular to the same line are parallel to each other.
4. If a transversal t intersects two lines l and m in distinct points A and B, then the lines l and m are said to make an intercept AB on t.
5. Two parallel lines make equal intercepts on all transversals perpendicular to them.
6. Equal Intercepts Property : If three or more parallel lines make equal intercepts on one transversal, then they make equal intercepts on any other transversal as well.
7. Proportional Intercepts Property : Three or more parallel lines intersecting any two transversals make intercepts on them in the same proportion.
8. We can divide a given line segment into a number of equal parts, using Equal Intercepts Property.
9. We can divide a given line segment in a given ratio internally, using Proportional Intercepts Property.

CHAPTER

10

SPECIAL TYPES OF QUADRILATERALS

10.1 Introduction

In Class VII, you have learnt about a quadrilateral, its sides, angles, diagonals and an important property regarding the sum of the four angles of a quadrilateral. In this Chapter, we shall learn about some special types of quadrilaterals such as trapeziums (trapezia), parallelograms, rectangles, rhombuses (rhombii) and squares. We shall also learn some of the properties of these quadrilaterals through activities.

10.2 Trapezium and Parallelogram

Look at the quadrilateral ABCD given in Fig. 10.1. In this quadrilateral, the opposite sides AB and DC are parallel. It is called a *trapezium*. Thus :

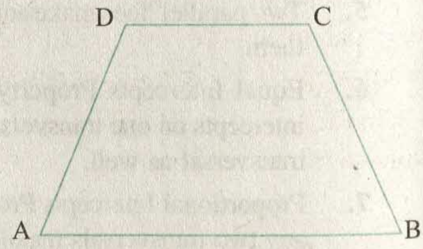


Fig. 10.1

A quadrilateral in which at least one pair of opposite sides are parallel is called a trapezium.

Now look at the quadrilateral EFGH given in Fig. 10.2. It is a trapezium because EF || HG. It is also a trapezium because EH || FG. Thus, it is a special trapezium in which opposite sides are parallel. Such a trapezium is called a *parallelogram*. Thus : A *parallelogram is a quadrilateral in which both the pairs of opposite sides are parallel.*

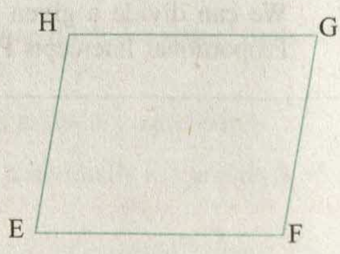


Fig. 10.2

Remark : You can easily see that every parallelogram is a trapezium but the converse is not true. In other words, not every trapezium is a parallelogram. For example, trapezium in Fig. 10.1 is not a parallelogram.

10.3 Rhombus, Rectangle and Square

Look at the parallelograms given in Fig. 10.3. In Fig 10.3 (i), ABCD is a parallelogram

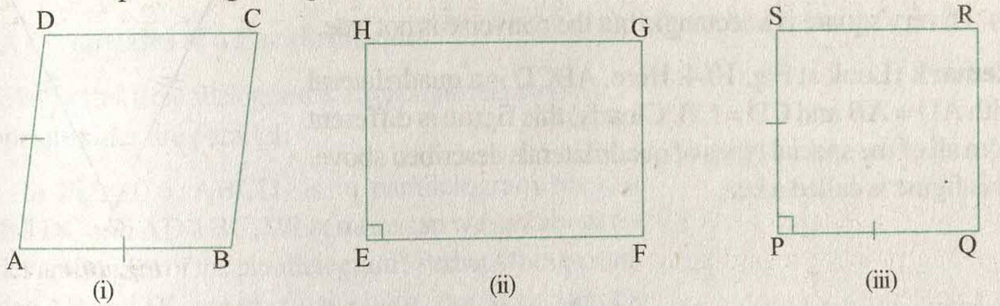


Fig. 10.3

with a pair of adjacent sides AB and AD equal. Such a parallelogram is called a *rhombus*. Thus :

A parallelogram in which a pair of adjacent sides are equal is called a rhombus.

In other words, a rhombus is a quadrilateral in which opposite sides are parallel, and a pair of adjacent sides equal. Note that parallelogram of Fig. 10.2 is not a rhombus. From properties of parallelogram we will see that all four sides of rhombus are equal.

In Fig. 10.3 (ii), EFGH is a parallelogram with $\angle E = 90°$. Such a parallelogram is called a *rectangle*. Thus :

A parallelogram with one angle a right angle is called a rectangle.

In other words, rectangle is a quadrilateral in which opposite sides in both the pairs are parallel and one angle a right angle. From properties of a parallelogram we will see that all four angles are right angles. Note that parallelogram of Fig. 10.2 is not a rectangle.

In Fig. 10.3 (iii), PQRS is a parallelogram with a pair of adjacent sides PQ and PS equal and $\angle P = 90°$. Such a parallelogram is called a *square*. Thus :

A parallelogram with a pair of adjacent sides equal and one angle a right angle is called a square.

In other words, a square is a quadrilateral in which opposite sides in both the pairs are parallel, one angle is a right angle and two adjacent sides are equal. All four sides of a square are equal. Note that rhombus of Fig. 10.3 (i) is not a square and rectangle of Fig. 10.3 (ii) is also not a square.

From the above, we can easily see that :

(i) Every rhombus is a parallelogram but the converse is not true.

(ii) Every rectangle is a parallelogram but the converse is not true.

(iii) Every square is a parallelogram but the converse is not true.

(iv) Every square is a rhombus but the converse is not true.

(v) Every square is a rectangle but the converse is not true.

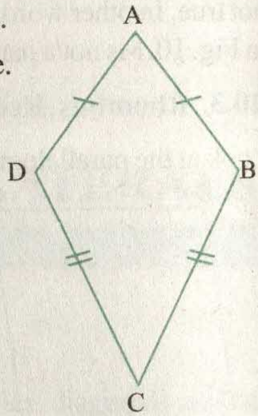


Fig. 10.4

Remark : Look at Fig. 10.4. Here, ABCD is a quadrilateral with AD = AB and CD = CB. Clearly, this figure is different from all of the special types of quadrilaterals described above. This figure is called a *kite*.

EXERCISE 10.1

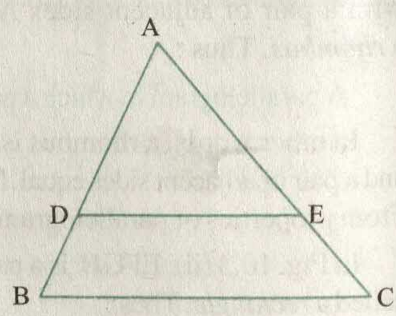


Fig. 10.5

1. In Fig. 10.5, DE || BC. What type of quadrilateral is BCED ?

2. How does a trapezium differ from a parallelogram?

3. ABCD is a parallelogram. What special name will you give it if the following additional facts are known?

(i) AB = AD.

(ii) $\angle$ DAB = 90°.

(iii) AB = AD and $\angle$ DAB = 90°.

4. ABCD is a trapezium in which AB || DC. If $\angle$ A = $\angle$ B = 40°, what are the measures of the other two angles?

5. The angles P, Q, R and S of a quadrilateral PQRS are in the ratio of 1 : 3 : 7 : 9.

(i) Find the measure of each angle.

(ii) Is PQRS a trapezium? Why?

(iii) Is PQRS a parallelogram? Why?

6. For each of the following statements, state whether the statement is true (T) or false (F) :

(i) Every rectangle is a parallelogram. (ii) Every square is a rectangle.

(iii) Every parallelogram is a rhombus. (iv) Every square is a rhombus.

(v) Every rectangle is a square.
(vi) Every parallelogram is a rectangle.
(vii) Every square is a parallelogram.
(viii) Every rhombus is a parallelogram.
(ix) Every rhombus is a square.
(x) Every parallelogram is a square.
(xi) Every parallelogram is a trapezium.
(xii) Every trapezium is a parallelogram.
(xiii) Every square is a trapezium.
(xiv) Every trapezium is a square.

10.4 Properties of a Parallelogram

We know that a parallelogram is a quadrilateral in which opposite sides are parallel.

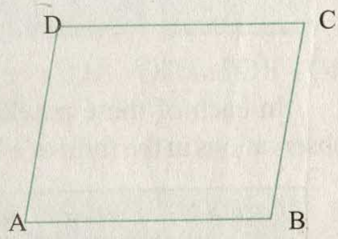


Fig. 10.6

In Fig. 10.6, ABCD is a parallelogram because AB || DC and AD || BC. What more can we say about the sides and angles of the parallelogram? Perhaps the opposite sides AB and DC are of equal length, and so also are the opposite sides AD and BC. Also it appears that the opposite angles A and C are of equal measure and so also are the opposite angles B and D. Let us examine.

Activity 1 : Draw a pair of parallel lines. Draw another pair of parallel lines intersecting the first pair of parallel lines. Label the parallelogram so formed as ABCD.

Draw two more parallelograms like this. Label each of these also as ABCD. Number these three parallelograms as 1, 2 and 3.

In each of these parallelograms, measure AB, BC, CD and DA. Write your observations in the form of a table as shown below :

Parallelogram	*First pair of opposite sides*			*Second pair of opposite sides*		
	AB	DC	AB – DC	AD	BC	AD – BC
1.						
2.						
3.						

What do you observe? You will observe that, in each case, the differences AB – DC and AD – BC are nearly zero. In other words, in appears that

AB = DC and AD = BC.

This activity illustrates the following proposition :

Opposite sides of a parallelogram are equal.

Activity 2 : Draw and number three parallelograms ABCD as in Activity 1 (Fig. 10.7).

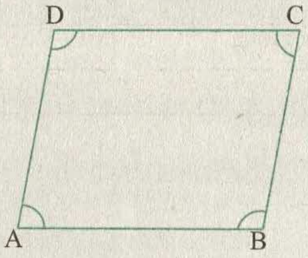


Fig. 10.7

In each of these parallelograms, measure ∠A, ∠C, ∠B and ∠D. Write your observations in the form of a table as shown below :

Parallelogram	*First Pair of Opposite Angles*			*Second Pair of Opposite Angles*		
	∠A	∠C	∠A – ∠C	∠B	∠D	∠B – ∠D
1.						
2.						
3.						

What do you observe? You will observe that, in each case, the differences ∠A – ∠C and ∠B – ∠D are nearly zero. In other words, it appears that ∠A = ∠C and ∠B = ∠D.

This activity illustrates the following proposition :

Opposite angles of a parallelogram are equal.

Look at Fig. 10.8 in which the two diagonals AC and BD of parallelogram ABCD intersect each other at the point O. It appears that O is the mid-point of AC as well as of BD. Let us examine.

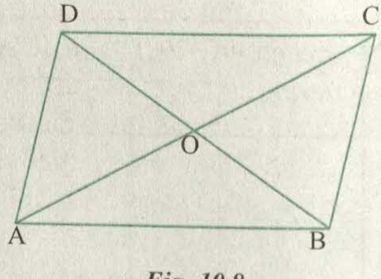


Fig. 10.8

Activity 3 : Draw a parallelogram ABCD as in Activity 1. Join AC and BD and let O be the point of intersection. Draw two more parallelograms. Label each of these parallelograms

as ABCD. In each parallelogram, join AC and BD and mark the point of intersection as O. Number the three parallelograms as 1, 2 and 3.

In each figure, measure OA, OC, OB and OD. Find the differences OA – OC and OB – OD. Write your observations in the form of a table as shown below :

Parallelo-gram	*First Diagonal* AC			*Second Diagonal* BD		
	OA	OC	OA – OC	OB	OD	OB – OD
1.						
2.						
3.						

What do you observe? You will observe that, in each case, OA – OC and OB – OD are nearly zero. Thus, in each case, it appears that

OA = OC and OB = OD.

This illustrates the following proposition :

Diagonals of a parallelogram bisect each other.

We now summarise the above observations :

In a parallelogram,

(i) opposite sides are equal,

(ii) opposite angles are equal, and

(iii) diagonals bisect each other.

We now describe one more activity to illustrate the above three properties of a parallelogram.

Activity 4 : Take a cardboard sheet and draw a parallelogram ABCD on it. Join AC and BD to intersect at the point O. With the help of a tracing paper, cut out a parallelogram congruent to the first parallelogram on another cardboard sheet. Label it as ABCD again. Let its diagonals AC and BD intersect at the point O. Now place the second parallelogram over the first by fixing a pin or nail at the point O so that both the parallelograms cover each other completely under the correspondence ABCD ↔ ABCD (Fig. 10.9).

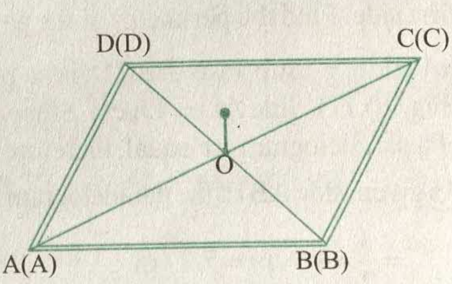


Fig. 10.9

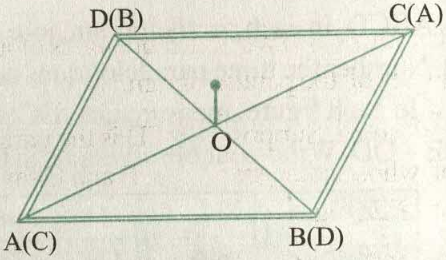


Fig. 10.10

Now let us rotate the upper parallelogram about the point O so that its vertex A comes to the position of vertex C of the other parallelogram (Fig. 10.10). What happens to the positions of vertices B, C and D of the rotating parallelogram? You will observe that B comes to the position of vertex D of the other parallelogram. Similarly, C comes to the position of vertex A and D comes to the position of vertex B of the other parallelogram. You can further note that in this position the two parallelograms cover each other completely. Thus,

We may say that

(i) AB = CD and AD = BC,

(ii) $\angle A = \angle C$ and $\angle B = \angle D$, and

(iii) OA = OC and OB = OD.

In other words, we have verified again that the

(i) opposite sides of a parallelogram are equal,

(ii) opposite angles of a parallelogram are equal, and

(iii) diagonals of a parallelogram bisect each other.

Let us now take some examples to illustrate these results.

Example 1 : One side of a parallelogram is 4.8 cm and the other side is $\frac{3}{2}$ times this first side. Find the perimeter of the parallelogram.

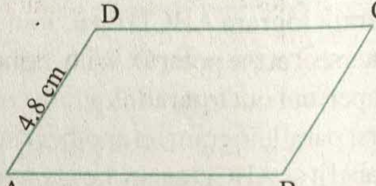


Fig. 10.11

Solution : Suppose ABCD is a parallelogram (Fig. 10.11). Side AD = 4.8 cm. Since opposite sides of a parallelogram are equal, therefore BC = 4.8 cm.

As given, side AB of the parallelogram

$$= \frac{3}{2} \times 4.8 \text{ cm} = 7.2 \text{ cm}$$

Therefore, the opposite side DC = 7.2 cm

Hence, perimeter of the parallelogram = 4.8 cm + 7.2 cm + 4.8 cm + 7.2 cm

= 24 cm

Example 2 : Two adjacent angles of a parallelogram are in the ratio of 1 : 2. Find all the angles of the parallelogram.

Solution : Suppose ABCD is the parallelogram, two of whose adjacent angles, B and C say, are in the ratio of 1 : 2 (Fig. 10.12).

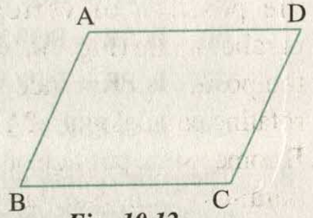


Fig. 10.12

$\angle B + \angle C = 180°$ (Interior angles on the same side of the transversal)

But it is given that

$$\angle B : \angle C = 1 : 2$$

Let $\angle B = x$. Therefore, $\angle C = 2x$.

Hence, $x + 2x = 180°$

Therefore, $\angle B = x = \frac{1}{3} \times 180° = 60°$

and $\angle C = 2x = \frac{2}{3} \times 180° = 120°$

Hence, $\angle D = \angle B = 60°$ and $\angle A = \angle C = 120°$ (Opposite angles of a parallelogram are equal)

EXERCISE 10.2

1. Lengths of two adjacent sides of a parallelogram are 4 cm and 3 cm. Find its perimeter.
2. One angle of a parallelogram is of measure 70°. Find the measures of the remaining angles of the parallelogram.
3. Two adjacent angles of a parallelogram are equal. Find the measure of each angle of the parallelogram.
4. The ratio of two sides of a parallelogram is 3 : 5 and its perimeter is 48 cm. Find the sides of the parallelogram.
   [***Hint :*** Let two sides be $3x$ and $5x$.]
5. Two adjacent angles of a parallelogram are in the ratio of 2 : 3. Find all the angles of the parallelogram.
6. The perimeter of a parallelogram is 150 cm. One of its sides is greater than the other by 25 cm. Find the lengths of all the sides of the parallelogram.

7. PR is a diagonal of the parallelogram PQRS (Fig. 10.13).

(i) Is PS = RQ? Why?

(ii) Is SR = PQ? Why?

(iii) Is PR = RP? Why?

(iv) Is Δ PSR $\cong$ Δ RQP? Why?

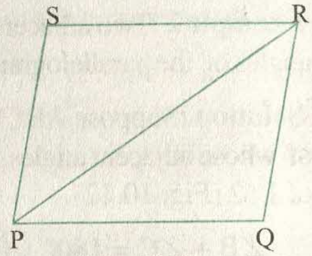


Fig. 10.13

8. The point of intersection of the diagonals of a quadrilateral divides one diagonal in the ratio 2 : 3. Can it be a parallelogram? Why?

9. Diagonals of a parallelogram ABCD intersect at the point O (Fig. 10.14). XY is a line segment passing through O such that X lies on AD and Y lies on BC. Give reasons for each of the following statements:

(i) OB = OD.

(ii) $\angle$ OBY = $\angle$ ODX .

(iii) $\angle$ BOY = $\angle$ DOX.

(iv) Δ BOY $\cong$ Δ DOX.

Now state whether or not XY is bisected at O.

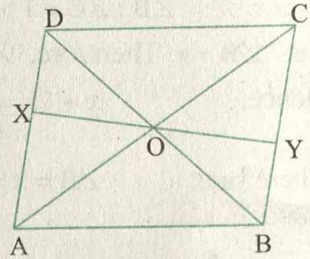


Fig. 10.14

10. In Fig. 10.15, ABCD is a parallelogram, CE bisects $\angle$ C and AF bisects $\angle$ A. Give reasons for each of the following statements :

(i) $\angle$ BAD = $\angle$ BCD.

(ii) $\angle \text{FAB} = \frac{1}{2} \angle \text{BAD}$.

(iii) $\angle \text{DCE} = \frac{1}{2} \angle \text{BCD}$.

(iv) $\angle$ FAB = $\angle$ DCE.

(v) $\angle$ DCE = $\angle$ CEB.

(vi) $\angle$ CEB = $\angle$ FAB.

(vii) CE $\parallel$ FA.

(viii) AE $\parallel$ FC.

(ix) AECF is a parallelogram.

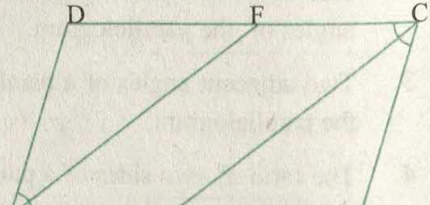


Fig. 10.15

10.5 Properties of a Rhombus

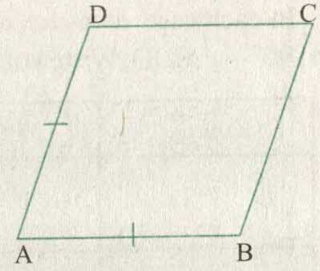


Fig. 10.16

Recall that rhombus is a parallelogram having a pair of adjacent sides equal. In Fig. 10.16, ABCD is a rhombus, i.e., a parallelogram with AB = AD.

Now from the properties of a parallelogram, we have :

(i) AB = DC and AD = BC

(ii) $\angle A = \angle C$ and $\angle B = \angle D$

Since AB = AD, we can easily conclude from (i) above that AB = BC = CD = AD. In other words, *all the sides of a rhombus are equal.*

Now let O be the point of intersection of the diagonals AC and BD of a rhombus ABCD (Fig. 10.17). From the properties of a parallelogram, we know that

OA = OC and OB = OD.

What can we say about the angles made by the diagonals AC and BD at their point of intersection O? To answer this question, let us perform the following activity :

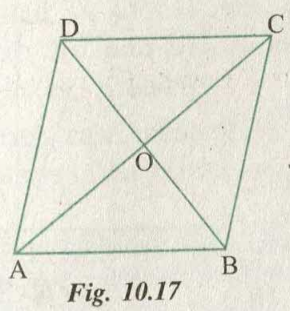


Fig. 10.17

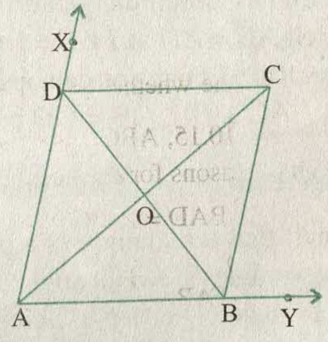


Fig. 10.18

Activity 5 : Draw $\angle XAY$. From its arms AX and AY, cut AD and AB respectively such that AD = AB.

Through D, draw a line parallel to AB and through B, draw a line parallel to AD to intersect each other at C (Fig. 10.18). Clearly, we obtain a parallelogram ABCD with AB = AD, i.e., a rhombus ABCD. Join AC and BD to intersect each other at O.

Draw two more rhombuses with different values of $\angle DAB$ and AB. Label the rhombuses so formed as ABCD. As before, label the point of intersection of the diagonals AC and BD as O. Number these rhombuses as 1, 2 and 3.

In each case, measure $\angle$BOC and $\angle$COD and find the differences $90° - \angle$BOC and $90° - \angle$COD. Write your observations in the form of a table as given below :

Rhombus	$\angle$ BOC	$90° - \angle$ BOC	$\angle$ COD	$90° - \angle$ COD
1. 2.				
3.				

What do you observe? You will observe that the differences $90° - \angle$BOC and $90° - \angle$COD, in each case, are nearly zero. Thus, it appears that

$\angle BOC = \angle COD = 90°$

Hence, $\angle DOA = \angle BOA = 90°$ (By the property of vertically opposite angles)

This activity illustrates the following proposition :

Diagonals of a rhombus bisect each other at right angles.

Remark : Recall that quadrilateral ABCD in which AB = AD and BC = DC is called a kite (Fig. 10.19). Here you can observe that

$AC \perp BD$, $OB = OD$ but $OA \neq OC$.

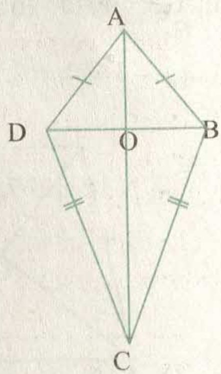


Fig. 10.19

10.6 Properties of a Rectangle

Recall that a rectangle is a parallelogram one of whose angles is a right angle. In Fig. 10.20, ABCD is a rectangle, i.e., a parallelogram with $\angle A = 90°$.

Since ABCD is a parallelogram, therefore

$AB = DC$, $AD = BC$,

$\angle A = \angle C$ and $\angle B = \angle D$.

Now AB || DC and AD is a transversal to these lines.

Therefore, $\angle A + \angle D = 180°$ (Interior angles on the same side of the transversal)

But $\angle A = 90°$

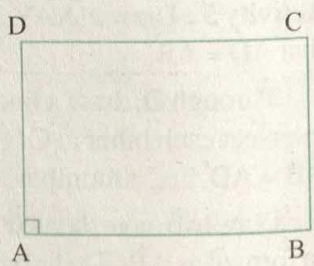


Fig. 10.20

Therefore, $\angle D = 90°$

Hence, $\angle B = 90°$ and $\angle C = 90°$ ($\angle B = \angle D$ and $\angle A = \angle C$).

Hence, *each angle of a rectangle is a right angle*.

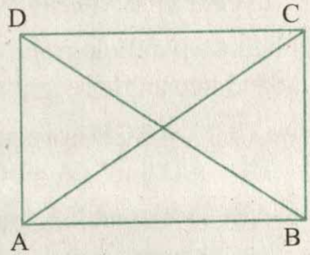


Fig. 10.21

Now, let us consider diagonals AC and BD (Fig. 10.21). They appear to be of the same length. Let us examine.

Activity 6 : Draw a right angle $\angle XAY$ and from the arms AX and AY, cut AD = 3 cm say and AB = 5 cm say respectively.

Through D and B respectively, draw lines parallel to AB and AD to intersect at C (Fig. 10.22). We obtain a rectangle ABCD. Join AC and BD.

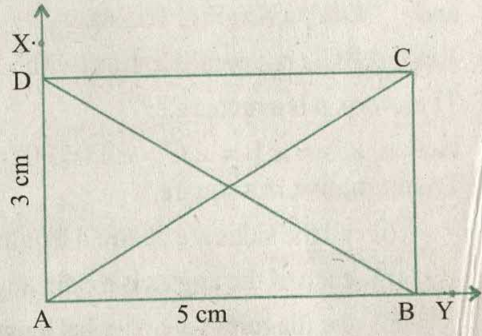


Fig. 10.22

Draw two more rectangles with different lengths AB and AD. Label the rectangles so formed as ABCD and draw their diagonals AC and BD. Number these rectangles as 1, 2 and 3.

In each case, measure AC and BD and find the difference AC – BD. Write your observations in the form of a table as given below:

Rectangle	*Diagonal* AC	*Diagonal* BD	AC – BD
1.			
2.			
3.			

What do you observe? You will observe that the difference AC – BD, in each case, is nearly zero.

Thus, it appears that AC = BD.

This activity illustrates the following proposition:

Diagonals of a rectangle are equal.

10.7 Properties of a Square

Recall that a parallelogram with a pair of adjacent sides equal and one angle a right angle is called a square.

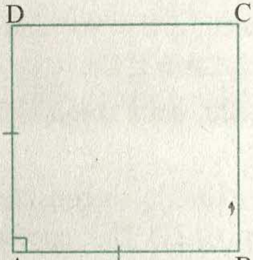


Fig. 10.23

In Fig. 10.23, ABCD is a square, i.e., it is a parallelogram with AB = AD and $\angle A = 90°$.

Now ABCD is a parallelogram with AB = AD.

Therefore, ABCD is a rhombus.

Hence, AB = BC = DC = AD,

$AC \perp BD$, OA = OC

and OB = OD (Fig. 10.24).

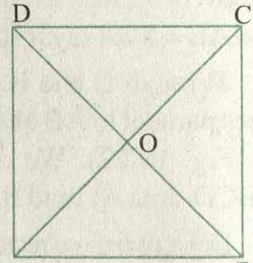


Fig. 10.24

Also, ABCD is a parallelogram with $\angle A = 90°$

Therefore, it is a rectangle.

Hence, $\angle A = \angle B = \angle C = \angle D = 90°$ and AC = BD.

To summarise, in a square,

(i) all the sides are of equal length.

(ii) each of the angles is a right angle.

(iii) the diagonals are of equal length.

(iv) the diagonals bisect each other at right angles.

We now take some examples to illustrate these properties.

Example 3 : Lengths of the diagonals AC and BD of a rhombus are 6 cm and 8 cm respectively. Find the length of each side of the rhombus.

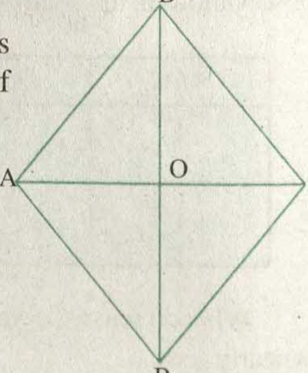


Fig. 10.25

Solution : Let the diagonals AC and BD of the rhombus bisect each other at O (Fig. 10.25).

Now AC = 6 cm and BD = 8 cm.

Therefore, $OA = \frac{6}{2}$ cm = 3 cm and

$OB = \frac{8}{2}$ cm = 4 cm (Diagonals bisect each other)

Now $\angle AOB = 90°$ (Diagonals of a rhombus are at right angles)

Therefore, $AB^2 = OA^2 + OB^2$ (Pythagoras Theorem)

$= 3^2 + 4^2 = 9 + 16 = 25$

or $AB = \sqrt{25} = 5$

i.e., length of each side of the rhombus is 5 cm.

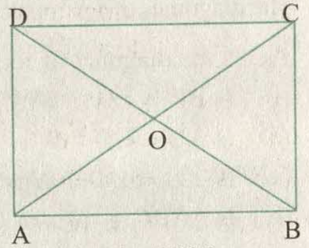


Fig. 10.26

Example 4 : Diagonals AC and BD of a rectangle ABCD intersect each other at a point O (Fig. 10.26). If OA = 5 cm, find AC and BD.

Solution : OA = 5 cm

Therefore,

AC = 2 OA = 2 × 5 cm = 10 cm (Diagonals bisect each other)

Also, AC = BD (Diagonals of a rectangle are equal)

Therefore, BD = 10 cm

EXERCISE 10.3

1. Which of the following statements are true for a rhombus?

(i) It has two pairs of parallel sides.

(ii) It has two pairs of equal angles.

(iii) It has only two pairs of equal sides.

(iv) Two of its angles are right angles.

(v) Its diagonals bisect each other at right angle.

(vi) Its diagonals are equal and perpendicular to each other.

(vii) It has all its sides of equal length.

2. Which of the following statements are true for a rectangle?

(i) It has two pairs of opposite sides of equal length.

(ii) It has all its sides of equal length.

(iii) Its diagonals are equal.

(iv) Its diagonals bisect each other.

(v) Its diagonals are perpendicular to each other.

(vi) Its diagonals are equal and perpendicular to each other.

(vii) Its diagonals are perpendicular and bisect each other.

(viii) Its diagonals are equal and bisect each other.

(ix) Its diagonals are equal, perpendicular and bisect each other.

(x) All of its angles are equal.

3. Repeat Question 2 above for a square in place of a rectangle.

4. The diagonals of a parallelogram are not perpendicular to each other. Is it a rhombus? Why?

5. AC is the diagonal of rectangle ABCD.

(i) Is BC = DA ? Why?

(ii) Is AB = CD ? Why?

(iii) Is $\angle B = \angle D$? Why?

(iv) Is $\Delta ABC \cong \Delta CDA$? By which congruence condition?

6. ABCD is a rhombus and its diagonals intersect each other at O (Fig. 10.27).

(i) Is OB = OD? Why?

(ii) Is BC = DC ? Why?

(iii) Is Δ BOC $\cong$ Δ DOC? By which congruence condition?

(iv) Is $\angle BCO = \angle DCO$? Why?

(v) Is Δ BAO $\cong$ Δ DAO? By which congruence condition?

(vi) Is $\angle BAO = \angle DAO$? Why?

(vii) Does diagonal AC of the rhombus bisect $\angle A$ and $\angle C$? Why?

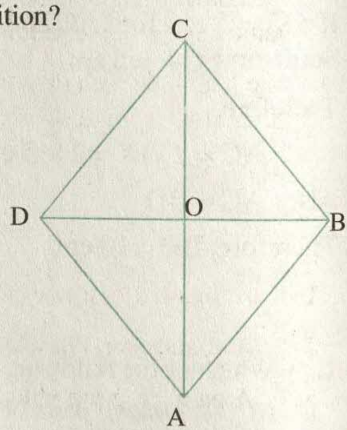


Fig. 10.27

7. Diagonal AC of a rhombus ABCD is equal to one of its sides BC (Fig. 10.28). Find all the angles of the rhombus.

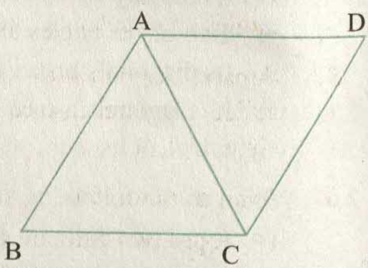


Fig. 10.28

8. A quadrilateral shaped window frame has one diagonal longer than the other. Is the window frame of the shape of a rectangle? Why ?

9. In Fig. 10.29, ABCD is a rectangle. BM and DN are perpendicular to AC from B and D respectively.

(i) Is AB = CD ? Why?

(ii) Is $\angle BMA = \angle DNC$? Why?

(iii) Is $\angle BAM = \angle DCN$? Why?

(iv) Is Δ BMA $\cong$ Δ DNC ? By which congruence condition?

(v) Is BM = DN ? Why?

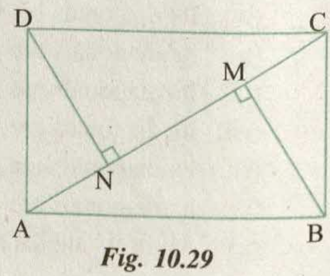


Fig. 10.29

10. The diagonals of a quadrilateral are perpendicular to each other. Is such a quadrilateral always a rhombus? If your answer is *No*, draw a figure to justify your answer.

11. Diagonals of a rhombus are equal. Is this rhombus also a square?

12. Diagonals of a quadrilateral are equal. Is such a quadrilateral always a rectangle? If your answer is *No*, draw a figure to justify your answer.

13. The diagonals of a quadrilateral are of lengths 10 cm and 24 cm. If the diagonals bisect each other at right angles, find the length of each side of the quadrilateral. What special name can you give to this quadrilateral?

14. The length of each diagonal of a quadrilateral is 12 cm. The diagonals also bisect each other at right angles. What special name can you give to this quadrilateral ?

Things to Remember

1. A quadrilateral two of whose opposite sides are parallel is called a trapezium.

2. A quadrilateral in which opposite sides are parallel is called a parallelogram.

3. A parallelogram with a pair of adjacent sides equal is called a rhombus. In fact, all the sides of a rhombus are equal.

4. A parallelogram with one angle a right angle is called a rectangle. In fact, all the angles of a rectangle are right angles.

5. A parallelogram with a pair of adjacent sides equal and one angle a right angle is called a square. In fact, all the sides of a square are equal and all its angles are right angles.

6. In a parallelogram,

(i) opposite sides are equal,

(ii) opposite angles are equal, and

(iii) diagonals bisect each other.

7. Diagonals of a rhombus bisect each other at right angles.

8. Diagonals of a rectangle are equal and bisect each other.

9. Diagonals of a square are equal and bisect each other at right angles.

CHAPTER

11

CONSTRUCTION OF QUADRILATERALS

11.1 Introduction

In Class VII, you have learnt about quadrilaterals. In the previous chapter, you have learnt about some special types of quadrilaterals and their properties. In this Chapter, we shall learn to construct some quadrilaterals with given measurements. Recall that in Class VII, you have learnt how to construct a triangle in the following simple cases :

(i) When two of its sides and the included angle are given.

(ii) When two of its angles and the included side are given.

(iii) When all three of its sides are given.

(iv) When the triangle is a right triangle and its hypotenuse and a side are given.

Observe that, in each of the above cases, knowledge of three specified parts of a triangle is sufficient to enable us to construct the triangle. Further, any two triangles constructed with the same measurements are congruent, i.e., exact copies of each other. In other words, given three suitable measures of a triangle, the triangle can be constructed uniquely.

How many measures do we need to construct a quadrilateral uniquely? We may imagine that, a knowledge of four specified parts of a quadrilateral should be sufficient to enable us to construct the quadrilateral uniquely. But this is not true. We explain it through an activity :

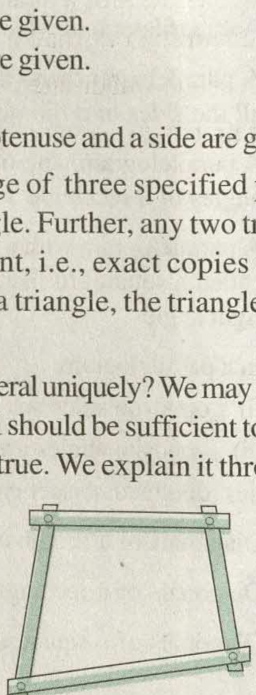


Fig. 11.1

Activity : Take four cardboard strips of suitable lengths, with holes at the ends. Hinge the strips at the ends to form a quadrilateral as shown in Fig. 11.1. Now try to change the shape of the quadrilateral by pressing at two opposite corners (vertices). You will find that you can easily change the shape and get a different

quadrilateral. Thus, it is possible to have two different quadrilaterals with the same measures of the four sides. In other words, even if we are able to construct a quadrilateral with four given sides, it is not unique.

Now take one more strip and join it to the quadrilateral made earlier, in the form of a diagonal as shown in Fig.11.2. Now try to change the shape of the quadrilateral. What do you observe? Now you are unable to do so. This shows that, in the case of a quadrilateral, it is necessary to have *at least* five parts (e.g., four sides and a diagonal as in this case) for constructing it uniquely. In what follows, we shall construct a quadrilateral in some simple cases. Of course, in each case, we shall need the measures of five specified parts of the quadrilateral. We shall explain these constructions through specific examples.

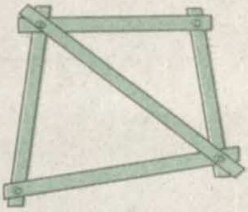


Fig. 11.2

11.2 Construction of a Quadrilateral when One Diagonal and Four Sides are Given

Given : Four sides of a quadrilateral ABCD as AB = 4 cm, BC = 6 cm, CD = 5 cm, AD = 5.5 cm and one diagonal AC = 8 cm.

To construct : A quadrilateral with these four sides and one diagonal.

We first draw a rough sketch of the quadrilateral ABCD and indicate the lengths of the four sides and the given diagonal [Fig. 11.3 (i)].

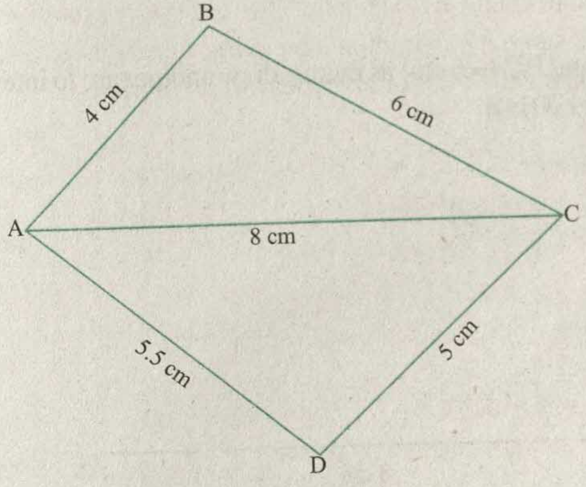


Fig. 11.3 (i)

Steps of construction :

1. Draw AC = 8cm [Fig. 11.3 (ii)].

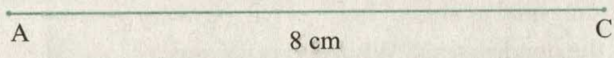


Fig. 11.3 (ii)

2. With A as centre and AB (= 4 cm) as radius, draw an arc [Fig. 11.3 (iii)].

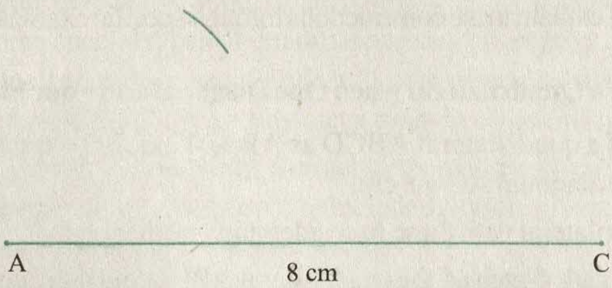


Fig. 11.3 (iii)

3. With C as centre and BC (= 6 cm) as radius, draw another arc to intersect the arc of Step 2 at B [Fig. 11.3 (iv)].

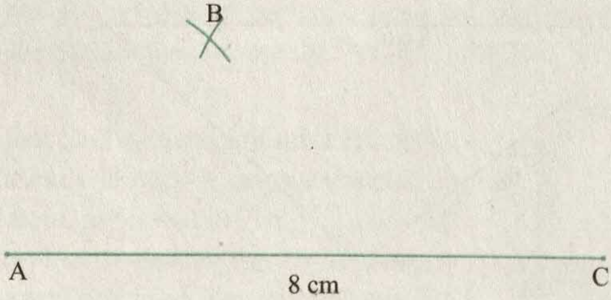


Fig. 11.3 (iv)

4. With A as centre and AD (= 5.5 cm), as radius draw an arc so that the arc and B are on opposite sides of AC [Fig. 11.3 (v)].

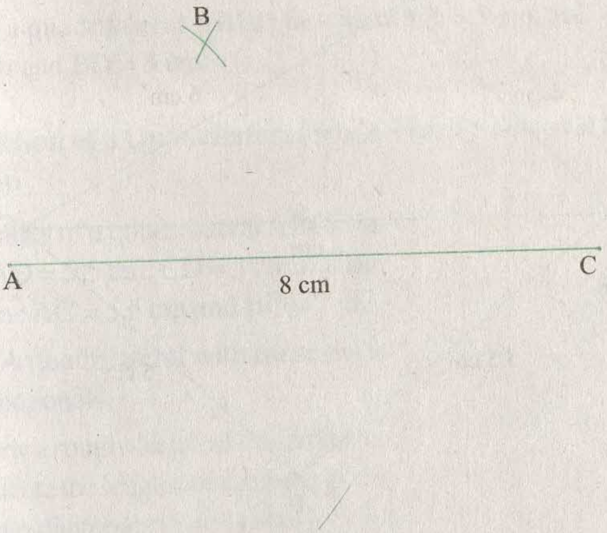


Fig. 11.3 (v)

5. With C as centre and CD (= 5 cm) as radius, draw another arc to intersect the arc of Step 4 at D [Fig. 11.3 (vi)].

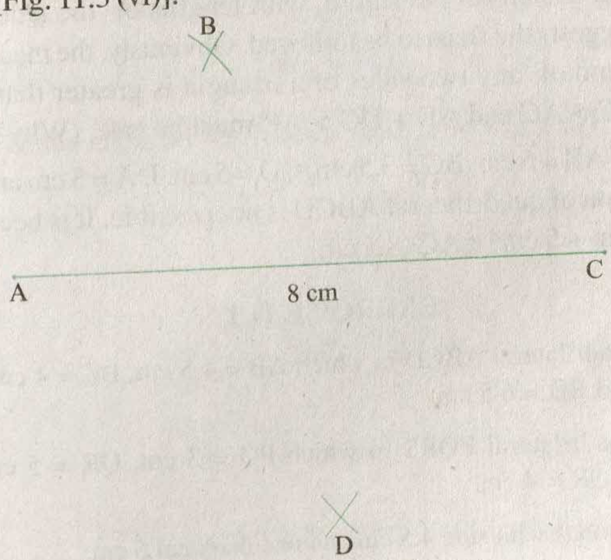


Fig. 11.3 (vi)

6. Join AB, BC, AD and CD [Fig. 11.3 (vii)].

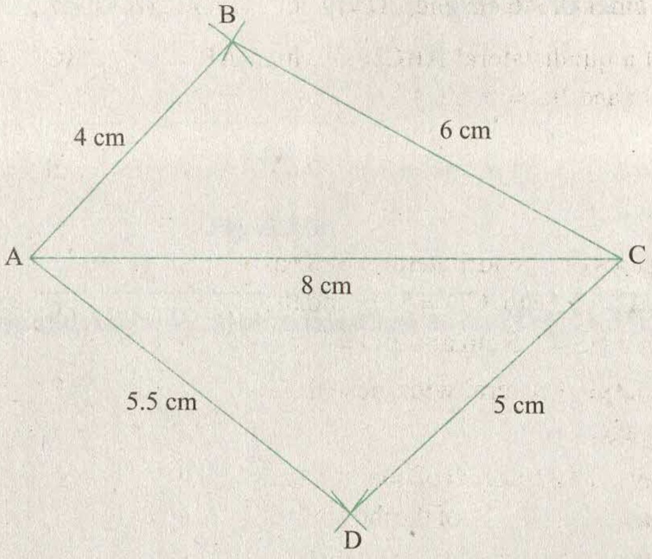


Fig. 11.3 (vii)

Then, ABCD is the required quadrilateral.

Remark : A rough sketch of the figure, with lengths of the sides and diagonal indicated, often suggests the steps to be followed. Obviously, the measurements must be such that the sum of any two sides of a triangle is greater than the third. For example, AB + BC > AC and AD + DC > AC must be true. (Why?)

For example, if AB = 5 cm, BC = 3.5 cm, CD = 5 cm, DA = 3 cm and AC = 8.5 cm, then the construction of quadrilateral ABCD is not possible. It is because of the fact that AD + CD (3 cm + 5 cm) < AC (8.5 cm).

EXERCISE 11.1

1. Construct a quadrilateral ABCD in which AB = 4.5 cm, BC = 4 cm, CD = 6.5 cm, DA = 3 cm and BD = 6.5 cm.

2. Construct a quadrilateral PQRS in which PQ = 3 cm, QR = 5 cm, QS = 5 cm, PS = 4 cm and SR = 4 cm.

3. Construct a rhombus with side 4.5 cm and one diagonal 6 cm.

4. Construct a parallelogram ABCD in which AB = 3.5 cm, BC = 4 cm and AC = 6.5 cm.

5. Is it possible to construct a quadrilateral ABCD in which AB = 3 cm, BC = 4 cm, CD = 5.5 cm, DA = 6 cm and BD = 9 cm ? If not, give reason.

6. Construct a quadrilateral ABCD in which AB = 5 cm, BC = 4 cm, AD = 3 cm, CD = 6 cm and BD = 5 cm.

11.3 Construction of a Quadrilateral when Three Sides and Both the Diagonals are Given

Given : Three sides of a quadrilateral ABCD as BC = 4.5 cm, AD = 5.5 cm, CD = 5 cm and the two diagonals as AC = 5.5 cm and BD = 7 cm.

To construct : A quadrilateral with these three sides and two diagonals.

We first draw a rough sketch of quadrilateral ABCD and indicate the lengths of the three given sides and the two diagonals [Fig. 11.4 (i)].

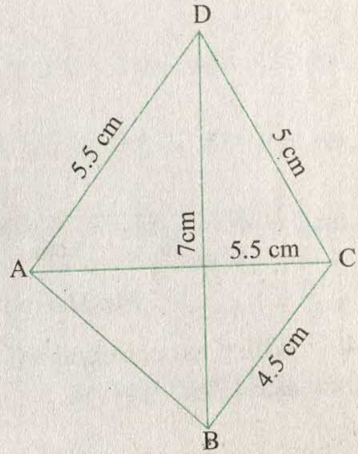


Fig. 11.4 (i)

Steps of construction :

1. Draw CD = 5 cm [Fig. 11.4 (ii)].

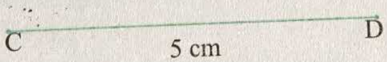


Fig. 11.4 (ii)

2. With C as centre and CB (= 4.5 cm) as radius, draw an arc [Fig. 11.4 (iii)].

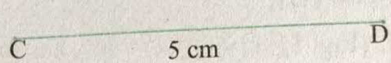


Fig. 11.4 (iii)

3. With D as centre and BD (= 7 cm) as radius, draw another arc to intersect the arc of Step 2 at B [Fig. 11.4 (iv)].

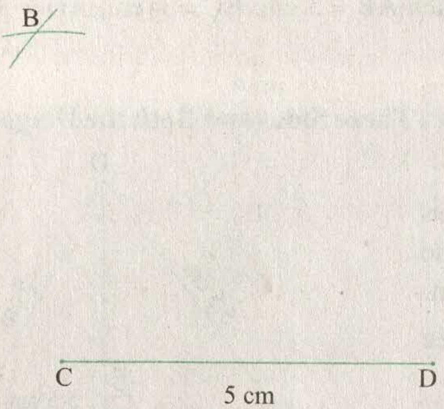


Fig. 11.4 (iv)

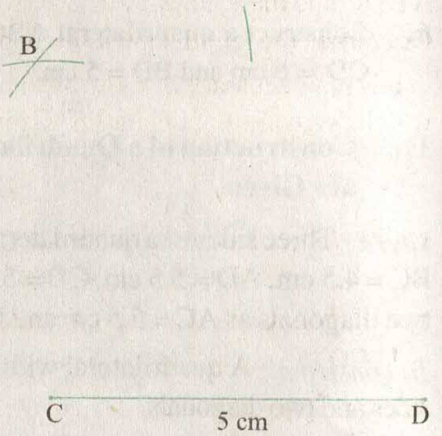


Fig. 11.4 (v)

4. With C as centre and AC (= 5.5 cm) as radius, draw an arc on the same side of CD as B [Fig. 11.4 (v)].

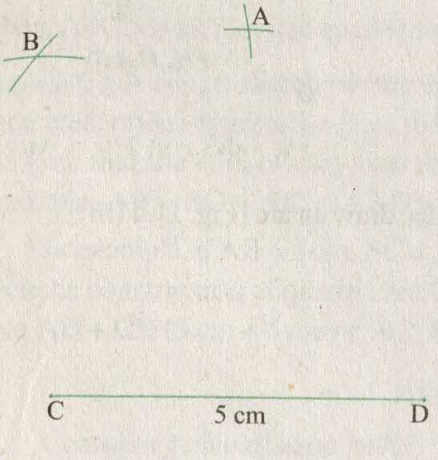


Fig. 11.4 (vi)

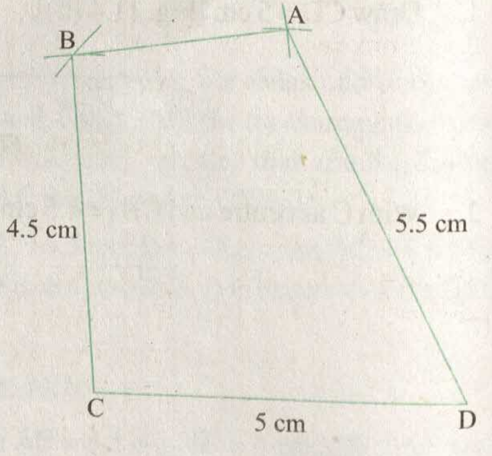


Fig. 11.4 (vii)

5. With D as centre and AD (= 5.5 cm) as radius, draw another arc to intersect the arc of Step 4 at A [Fig. 11.4 (vi)].

6. Join DA, AB and BC [Fig. 11.4 (vii)].
Then, ABCD is the required quadrilateral.

Remarks 1 : Making a rough sketch of the quadrilateral ABCD and indicating various lengths, we observe that complete measurements are known for drawing Δ ADC and Δ BDC and these have side CD as a common side. This suggests the steps of construction. Of course, the lengths must be such that CB + BD > CD and CA + AD > CD (Why?).

2. It is not necessary to show the diagonals. The required figure is quadrilateral ABCD which consists of only four line segments AB, BC, CD and DA.

EXERCISE 11.2

1. Construct a quadrilateral ABCD in which AB = 4 cm, BC = 3 cm, AD = 2.5 cm, AC = 4.5 cm and BD = 4 cm.
2. Construct a quadrilateral ABCD in which BC = 7.5cm, AC = AD = 6 cm, CD = 5 cm and BD = 10 cm.
3. Is it possible to construct a quadrilateral ABCD in which AB = 3cm, CD = 3 cm, DA = 7.5 cm, AC = 8 cm and BD = 4 cm? If not, give reason.
4. Construct a quadrilateral ABCD in which AB = 7 cm, AD = 6 cm, AC = 7cm, BD = 7.5 cm and BC = 4 cm.
5. Construct a quadrilateral ABCD in which AB = AD = 3cm, BC = 2.5 cm, AC = 4 cm and BD = 5 cm.
6. Construct a quadrilateral PQRS in which QR = 7.5 cm, RP = PS = 6 cm, RS = 5 cm and QS = 10 cm.

11.4 Construction of a Quadrilateral when Two Adjacent Sides and Three Angles are Given

Given : Two adjacent sides of a quadrilateral ABCD as AB = 3.5 cm, BC = 6.5 cm and its three angles as $\angle A = 75°$, $\angle B = 105°$ and $\angle C = 120°$.

To construct : A quadrilateral with these two adjacent sides and three angles.

We first draw a rough sketch of the quadrilateral ABCD and indicate the measures of the two given adjacent sides and three given angles [Fig. 11.5 (i)].

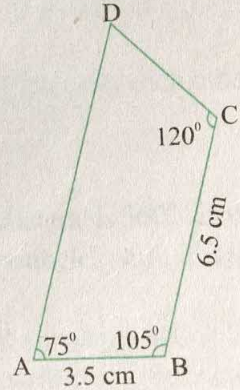


Fig. 11.5 (i)

Steps of construction :

1. Draw AB = 3.5 cm [Fig. 11.5 (ii)].

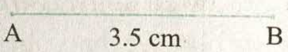


Fig 11.5 (ii)

2. At B, draw an angle XBA = 105° [Fig. 11.5 (iii)].

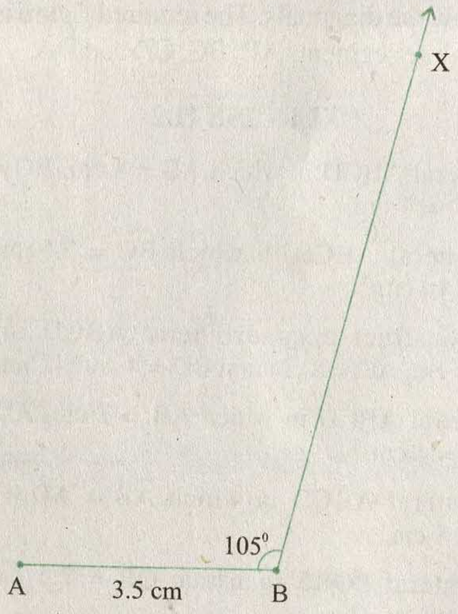


Fig 11.5 (iii)

3. From ray BX, cut BC = 6.5 cm [Fig. 11.5 (iv)].

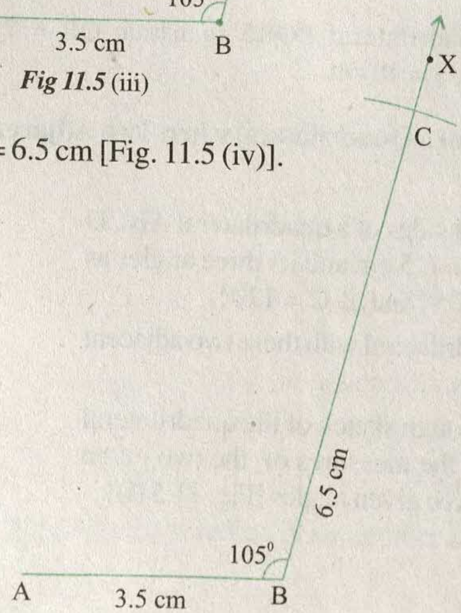


Fig 11.5 (iv)

4. At C, draw an angle YCB = 120° [Fig. 11.5 (v)].

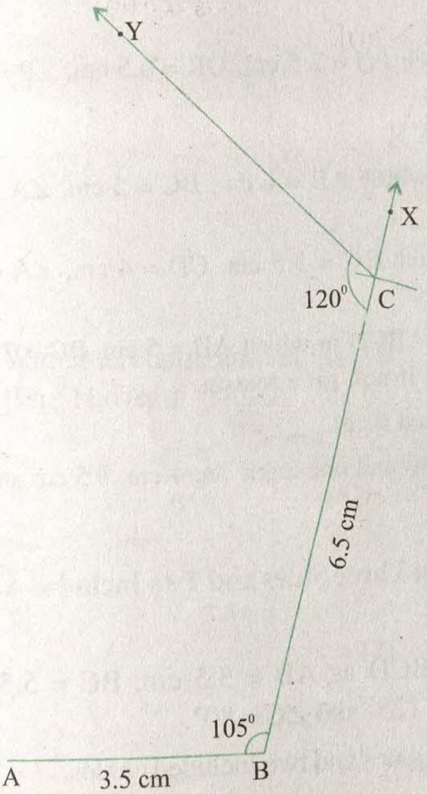


Fig. 11.5 (v)

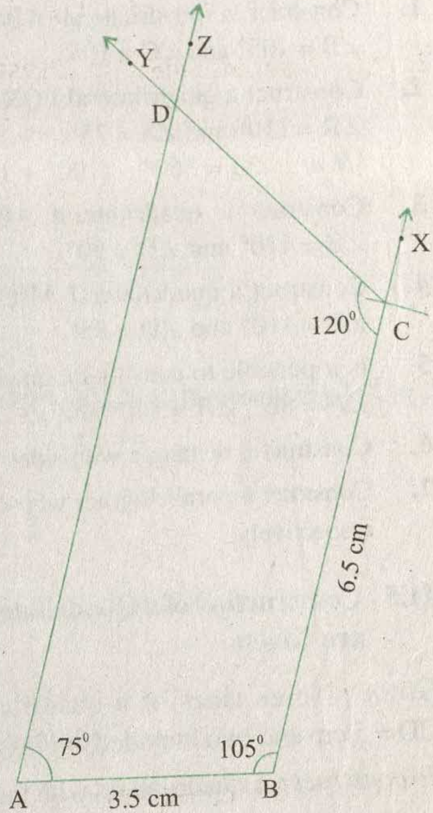


Fig. 11.5 (vi)

5. At A, draw an angle ZAB = 75° and let rays CY and AZ intersect each other at D [Fig. 11.5 (vi)].

Then, ABCD is the required quadrilateral.

Remark : We know that the sum of the four angles of a quadrilateral is 360°. Hence, for possible construction of the quadrilateral, the sum of the three angles ($\angle$A, $\angle$B and $\angle$C in this case) must be less than 360°.

For example, construction of a quadrilateral ABCD in which BC = 9.5 cm, $\angle A = 75°$, $\angle B = 150°$, AB = 6 cm and $\angle C = 140°$ is not possible. It is because of the fact that $\angle A + \angle B + \angle C\ (= 75° + 150° + 140°) > 360°$.

EXERCISE 11.3

1. Construct a quadrilateral ABCD in which AB = 5.5 cm, BC = 4 cm, $\angle A = 60°$, $\angle B = 105°$ and $\angle C = 105°$.
2. Construct a quadrilateral PQRS in which PQ = 3.5 cm, QR = 6.5 cm, $\angle P = 100°$, $\angle R = 110°$ and $\angle S = 75°$.
   [*Hint :* $\angle Q = 360° - (100° + 110° + 75°)$]
3. Construct a quadrilateral ABCD in which AB = 6 cm, BC = 5 cm, $\angle A = 55°$, $\angle B = 110°$ and $\angle D = 90°$.
4. Construct a quadrilateral ABCD in which BC = 5.5 cm, CD = 4 cm, $\angle A = 70°$, $\angle B = 110°$ and $\angle D = 85°$.
5. Is it possible to construct a quadrilateral ABCD in which AB = 5 cm, BC = 7.5 cm, $\angle A = 80°$, $\angle B = 140°$ and $\angle C = 145°$? If not, give reason.
6. Construct a rectangle with sides 4.5 cm and 6 cm.
7. Construct a parallelogram whose two sides and one angle are 4 cm, 5.5 cm and 70° respectively.

11.5 Construction of a Quadrilateral when Three Sides and Two Included Angles are Given

Given : Three sides of a quadrilateral ABCD as AB = 3.5 cm, BC = 5.5 cm, CD = 5 cm and two included angles as $\angle B = 125°$ and $\angle C = 80°$.

To construct : A quadrilateral with these three sides and two included angles.

We first draw a rough sketch of quadrilateral ABCD and indicate the measures of the three given sides and the two included angles [Fig. 11.6 (i)].

Steps of construction :

1. Draw BC = 5.5 cm [Fig. 11.6(ii)].

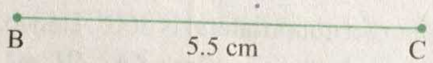


Fig. 11.6 (ii)

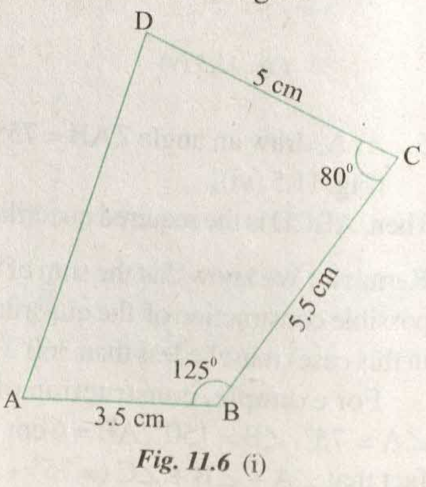


Fig. 11.6 (i)

2. Draw $\angle XBC = 125°$ [Fig. 11.6 (iii)].

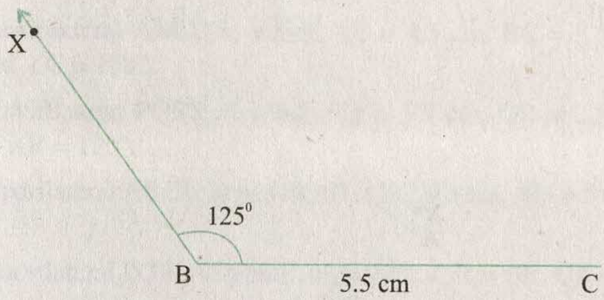


Fig. 11.6 (iii)

3. With B as centre and radius AB = 3.5 cm, draw an arc to intersect ray BX at A [Fig. 11.6 (iv)].

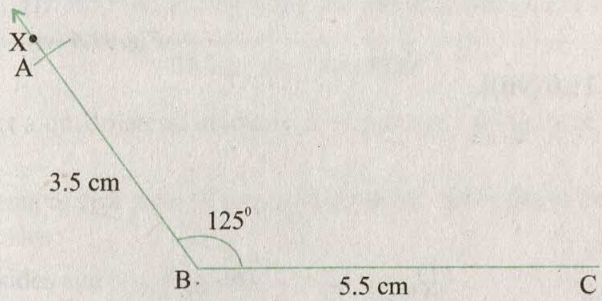


Fig. 11.6 (iv)

4. At C, draw $\angle YCB = 80°$ [Fig. 11.6 (v)].

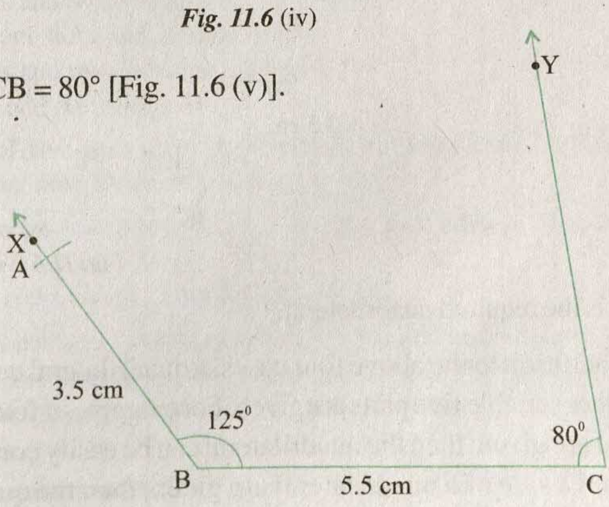


Fig. 11.6 (v)

5. With C as centre and radius CD = 5 cm, draw an arc to intersect ray CY at D [Fig. 11.6 (vi)].

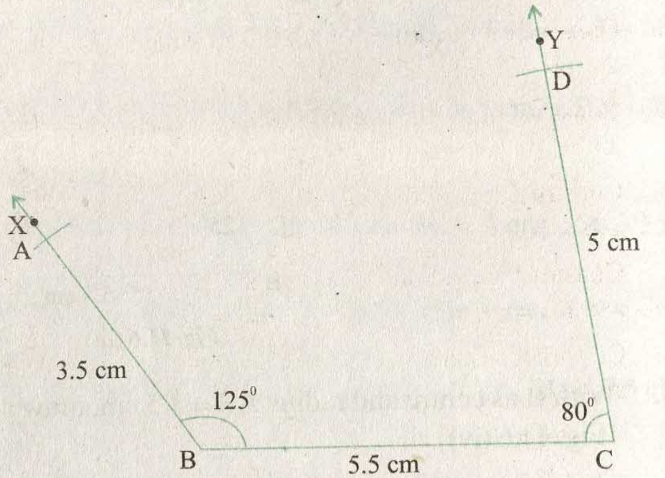


Fig. 11.6 (vi)

6. Join AD [Fig. 11.6 (vii)].

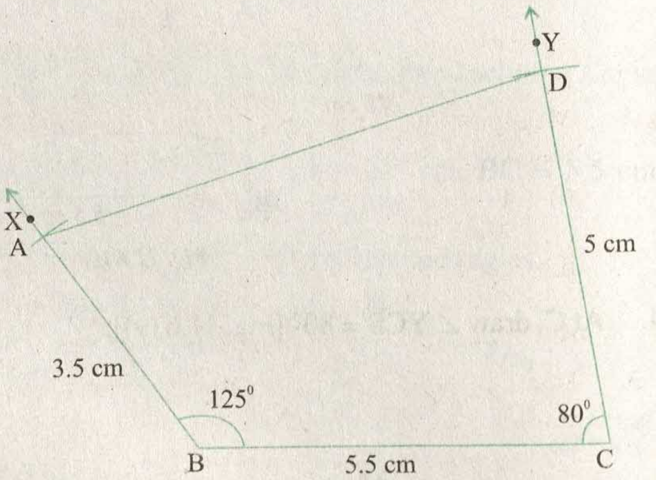


Fig. 11.6 (vii)

Then, ABCD is the required quadrilateral.

Remark : In addition to the above four cases, a quadrilateral can also be constructed when some other suitable five parts are given. For example, if four sides and an angle of a quadrilateral are given, then the quadrilateral can be easily constructed. However, if four angles and a side of a quadrilateral are given, then the quadrilateral cannot be constructed uniquely.

EXERCISE 11.4

1. Construct a quadrilateral ABCD in which AB = 4.5 cm, BC = 3.5 cm, CD = 5 cm, $\angle$ B = 45° and $\angle$C = 150°.
2. Construct a quadrilateral PQRS in which PQ = 3.5 cm, QR = 2.5 cm, RS = 4 cm, $\angle$Q = 75° and $\angle$R = 120°.
3. Construct a quadrilateral ABCD in which AB = BC = 3 cm, AD = 5 cm, $\angle$A = 90° and $\angle$B = 105°.
4. Construct a quadrilateral PQRS in which $\angle$Q = 45°, $\angle$ R = 90°, QR = 5 cm, PQ = 4 cm and RS = 3 cm.
5. Construct a quadrilateral ABCD in which AB = AD = 5 cm, CD = 5.5 cm, $\angle$A = 90° and $\angle$D = 120°.
6. Construct a trapezium ABCD in which AB || CD, AB = 8 cm, BC = 6 cm, CD = 4 cm and $\angle$B = 60°. [*Hint* : Find $\angle$C by using the fact that AB || CD.]

Things to Remember

1. To construct a quadrilateral uniquely, it is necessary to know at least five of its parts.
2. Measurements of five parts of a quadrilateral are sufficient to construct it in the following cases :
   (i) Four sides and one diagonal.
   (ii) Three sides and both diagonals.
   (iii) Two adjacent sides and three angles.
   (iv) Three sides and two included angles.
   (v) Four sides and one angle.
3. Measurements of five parts of a quadrilateral, in order to be sufficient for its construction, must also satisfy, wherever relevant,
   (i) triangle inequality property, i.e., sum of two sides is greater than the third side.
   (ii) angle sum property of a quadrilateral.
4. It is possible to construct a quadrilateral with other sufficient measurements (other than the above five simple cases), where less than five parts but some other relations between the parts are given (e.g. quadrilateral being a parallelogram or a rectangle, etc.).
5. It is always convenient and helpful to draw a rough sketch of the quadrilateral and indicate the given measurements.

CHAPTER

12

CIRCLES

12.1 Introduction

You are already familiar with circles and related concepts such as centre, radius, diameter, arc, semicircle, chord, segment of a circle, etc. from your earlier classes. In Class VII, you have also learnt the following two properties of circles :

1. Angle in a semicircle is a right angle.
2. Angles in the same segment of a circle are equal.

In this Chapter, we shall learn some more properties of circles. These properties are related to perpendiculars from the centre to a chord, angles subtended by chords and arcs at the centre or a point on the circle, and angles of a cyclic quadrilateral.

12.2 Perpendicular from the Centre to a Chord

Activity 1 : Draw a circle with centre O. Also, draw a chord AB. Draw OM $\perp$ AB such that M lies on AB (Fig. 12.1). Repeat this activity by drawing two more circles with different radii. For convenience, label the centre, chord and the perpendicular in each of these three cases as O, AB and OM, respectively. In other words, *label the figures similarly*. Number the circles as 1, 2 and 3.

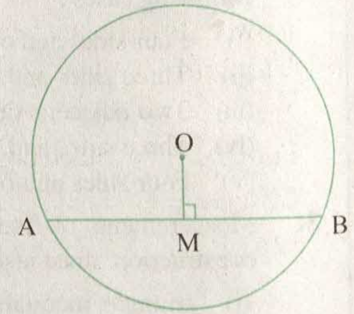


Fig. 12.1

In each case, measure AM, BM, and find the difference AM – BM. Record your observations in the form of a table as given below :

Circle	AM	BM	AM – BM
1.			
2.			
3.			

What do you observe? You will observe that in each case, the difference AM – BM is nearly zero. Thus, in all cases, it appears that AM = BM.

Activity 2 : Take a sheet of tracing paper and draw a circle with centre O on it. Also draw a chord AB of the circle [Fig. 12.2 (i)]. Now fold the circle over itself so that A falls on B. Press to form a crease along the line l [Fig. 12.2 (ii)]. Note that the line l passes through O and a part of chord AB falls along a part of itself so that the two parts completely cover each other.

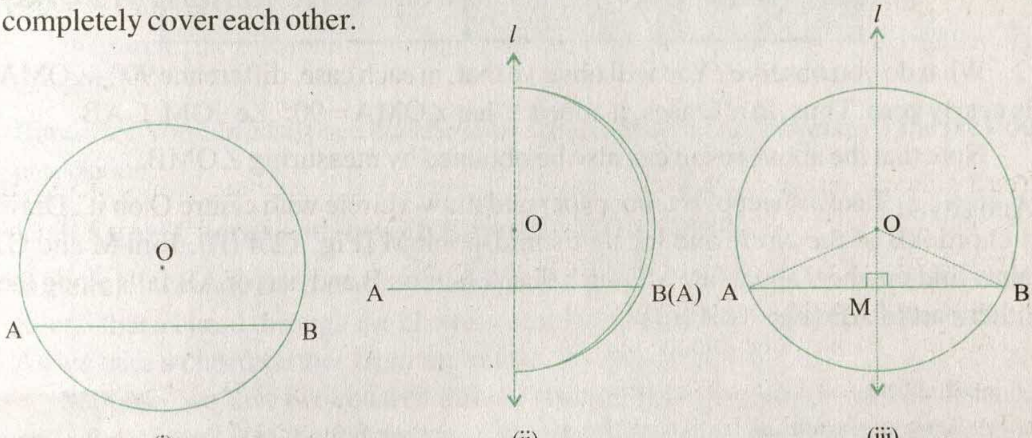


Fig. 12.2

Now unfold the paper and mark the point of intersection of AB and l as M [Fig. 12.2 (iii)]. Now since OM falls along itself and AM falls along BM, therefore,

$$\angle \text{ OMA} = \angle \text{ OMB} = 90°, \text{ i.e., OM} \perp \text{AB}.$$

Further, in the folded position, since AM coincides with BM, therefore, AM = BM.

Observe that in Fig 12.2(iii) OA = OB; being radii of the same circle. You can use this to show that triangles OAM and OBM are congruent. Hence AM = BM. Thus :

In a circle, perpendicular from the centre to a chord bisects the chord.

Activity 3 : Draw a circle with centre O. Also draw chord AB. Bisect AB at M and join O to M (Fig.12.3). Repeat this activity by drawing two more circles with different radii. Label the figures similarly. Number the circles as 1, 2 and 3.

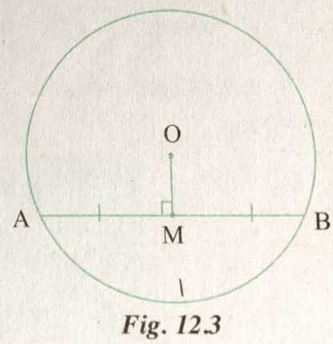


Fig. 12.3

Measure $\angle OMA$ and find the difference $90° - \angle OMA$ in each case. Record your observations in the form of a table as given below :

Circle	$\angle OMA$	$90° - \angle OMA$
1.		
2.		
3.		

What do you observe? You will observe that, in each case, difference $90° - \angle OMA$ is nearly zero. Thus, in all cases, it appears that $\angle OMA = 90°$, i.e., $OM \perp AB$.

Note that the above result can also be obtained by measuring $\angle OMB$.

Activity 4: Take a sheet of tracing paper and draw a circle with centre O on it . Draw a chord AB of the circle and locate its mid-point M [Fig. 12.4 (i)]. Join M and O. Now fold the sheet about line MO such that A falls on B and part of AB falls along the other part of AB [Fig. 12.4 (ii)].

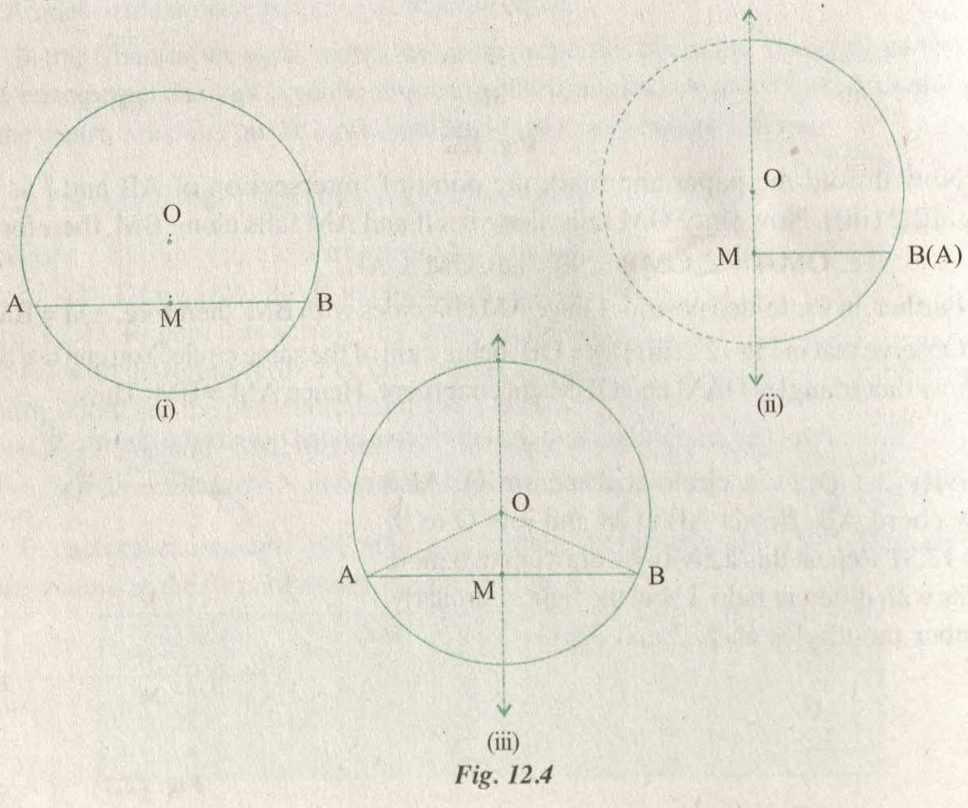


Fig. 12.4

Now unfold the sheet [Fig. 12.4 (iii)]. What do you observe? Does $\angle OMA$ fall along $\angle OMB$? You will observe that it does fall along $\angle OMB$. Thus,

$$\angle OMA = \angle OMB = 90^\circ$$

i.e., $OM \perp AB$

Observe that in figure 12.4 (iii) OA = OB. Hence the ΔOAB is isosceles and $\angle A = \angle B$. Thus, $\Delta OMA \cong \Delta OMB$. Hence $\angle OMA = \angle OMB$ is right angle.

In a circle, the line joining the mid-point of a chord to the centre is perpendicular to the chord.

Remark : You can easily see that the above proposition is the *converse* of the previous proposition.

12.3 Equal Chords and their Distances from the Centre

In a circle, we may have any number of chords. Some chords are longer than others. Recall that a chord through the centre, i.e., a diameter is the longest chord of a circle. As we take a chord farther from the centre, it gets shorter.

Suppose, we have two equal chords in a circle. What can we say about their distances from the centre? Are they at the same distance from the centre? On the other hand, suppose, we have two chords at the same distance from the centre. Are these chords equal? Let us perform some activities to answer these questions.

Activity 5 : Draw a circle with centre O and draw two chords AB and CD such that AB = CD as shown in Fig. 12.5. From O, draw $OM \perp AB$ and $ON \perp CD$ such that M and N lie on AB and CD, respectively. Repeat this activity by drawing two more circles with different centres and radii. Label the figures similarly. Number the circles as 1, 2 and 3.

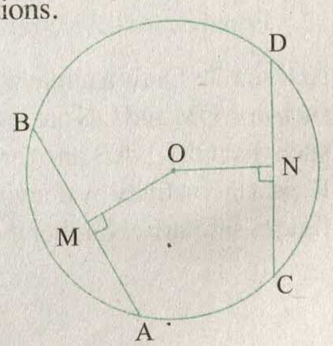


Fig. 12.5

In each case, measure OM and ON and find the difference OM – ON. Record your observations in the form of a table as given further.

Circle	OM	ON	OM – ON
1.			
2.			
3.			

What do you observe? You will observe that, in each case, difference OM – ON is nearly zero. Thus, in all cases, it appears that OM = ON.

Activity 6 : Take a sheet of tracing paper and draw a circle with centre O on it. Draw two equal chords AB and CD of the circle. From O, draw OM ⊥ AB and ON ⊥ CD such that M and N lie on AB and CD, respectively [Fig. 12.6 (i)]. Fold the sheet along a line passing through O such that A falls on C and B falls on D [Fig. 12.6 (ii)]. What happens to M? Does it fall on N? Yes, it does fall on N. Thus, OM = ON.

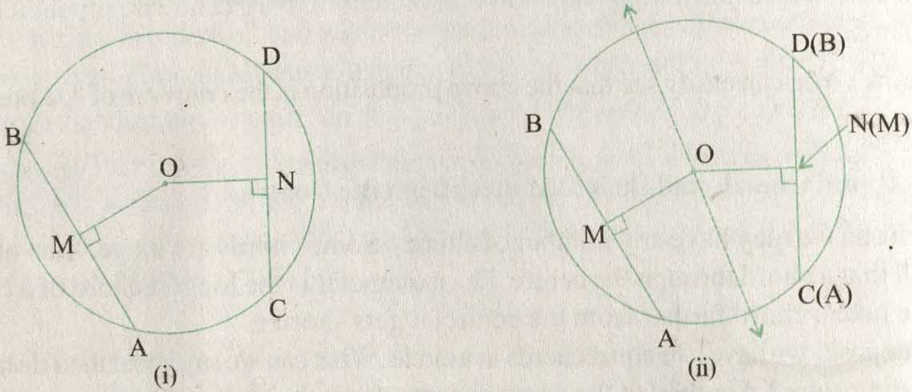


Fig. 12.6

The above two activities illustrate the following proposition:

Equal chords of a circle are equidistant from the centre.

Activity 7 : Draw a circle with centre O and draw two equal line segments OM and ON such that OM and ON are less than the radius of the circle. Through M, draw a chord AB such that OM ⊥ AB and through N, draw a chord CD such that ON ⊥ CD (Fig. 12.7). Repeat this activity by drawing two more circles with different centres and radii. Label the figures similarly. Number the circles as 1, 2 and 3.

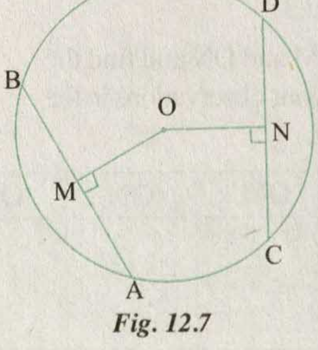


Fig. 12.7

Now measure AB and CD and find the difference AB – CD in each case. Record your observations in the form of a table as given below.

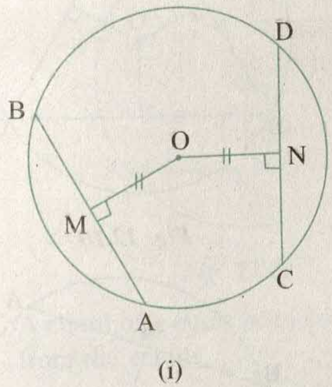
Circle	Chord AB	Chord CD	AB – CD
1.			
2.			
3.			

What do you observe? You will observe that, in each case, AB – CD is nearly zero. Thus, in all cases, it appears that

AB = CD, i.e., the chords are equal.

Activity 8 : Take a sheet of tracing paper and on it draw a circle with centre O. Now, draw two equal line segments OM and ON such that OM (= ON) is less than the radius of the circle. Through M and N, draw two chords AB and CD, respectively such that OM ⊥ AB and ON ⊥ CD [Fig. 12.8 (i)].

Fold the sheet along a line passing through O such that M falls on N [Fig. 12.8 (ii)]. What do you observe? You will observe that A falls on C and B falls on D. Thus, AB = CD.



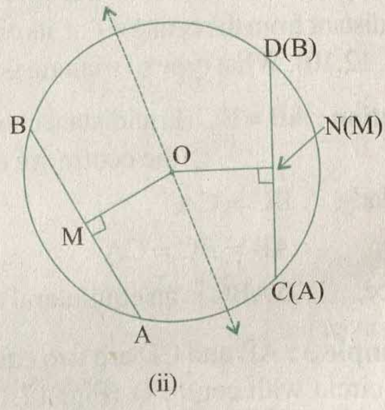


Fig. 12.8

The above two activities illustrate the following proposition :

Chords equidistant from the centre of a circle are equal.

Remark : Note that this proposition is the *converse* of the previous proposition.

We now take some examples to illustrate these propositions.

Example 1 : In a circle of radius 10 cm, a chord is at a distance 6 cm from the centre. Find the length of the chord.

Solution : Let AB be a chord of a circle of radius 10 cm at a distance 6 cm from the centre O.

Let OM $\perp$ AB (Fig. 12.9).

Thus, we have OA = 10 cm and OM = 6 cm.

Now from Pythagoras Theorem,

$$OA^2 = AM^2 + OM^2$$

or $$AM^2 = OA^2 - OM^2$$

$$= (10^2 - 6^2)\text{ cm}^2 = 64\text{ cm}^2$$

or $$AM = 8\text{ cm}$$

Now $AB = 2\,AM$ (Perpendicular from the centre bisects the chord)

i.e., $AB = 2 \times 8\text{ cm} = 16\text{ cm}$

Thus, the length of the required chord is 16 cm.

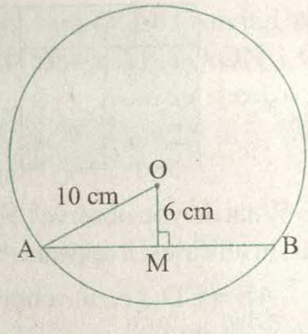


Fig. 12.9

Example 2 : Sides of a triangle ABC are equidistant from the centre O of its circumcircle (Fig. 12.10). What type of triangle is Δ ABC?

Solution : AB = BC (Equidistant chords from the centre are equal)

Similarly, BC = CA

Thus, AB = BC = CA

Hence, ΔABC is an equilateral triangle.

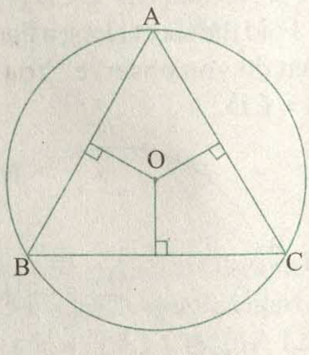


Fig. 12.10

Example 3 : AB and CD are two equal chords of a circle with centre O (Fig. 12.11). When produced, AB and CD meet at S outside the circle. OM $\perp$ AB and ON $\perp$ CD, where M and N lie on AB and CD, respectively. Give reasons for the following statements :

(i) OM = ON

(ii) Δ OMS $\cong$ Δ ONS

(iii) MS = NS

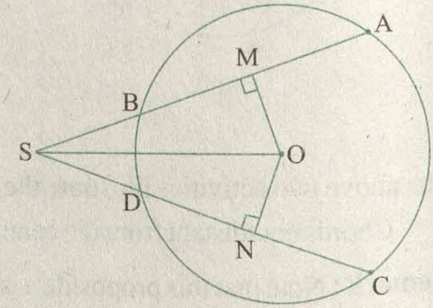


Fig. 12.11

(iv) AM = CN

(v) AS = CS

Solution: (i) OM = ON (Equal chords are equidistant from the centre O)

(ii) $\Delta OMS \cong \Delta ONS$ (By RHS as $\angle OMS = \angle ONS = 90°$, OS = OS and OM = ON)

(iii) MS = NS [Corresponding parts of congruent triangles (CPCT)]

(iv) AM = CN (M and N are mid-points of equal chords AB and CD)

(v) AS = CS (AS = AM + MS and CS = CN + NS)

EXERCISE 12.1

1. AB is a chord of a circle of radius 5 cm with centre O (Fig. 12.12). If OM $\perp$ AB and AB = 8 cm, find the length OM.

2. AB is a chord of the circle with centre O and radius 13 cm (Fig. 12.13). If OM $\perp$ AB and OM = 5 cm, find the length of the chord AB.

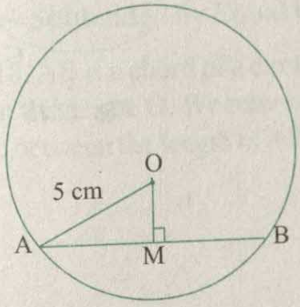


Fig. 12.12

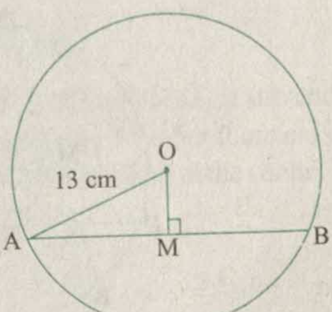


Fig. 12.13

3. A chord of a circle of radius 7.5 cm with centre O is of length 9 cm. Find its distance from the centre.

4. In a circle of radius 13 cm, a chord is drawn at a distance of 12 cm from the centre. Find the length of the chord.

5. A chord of a circle is of length 6 cm and is at a distance of 4 cm from the centre. Find the radius of the circle.

6. A, B and C are three points of a circle whose centre is not indicated. How will you locate the centre?

[*Hint :* If M is the mid-point of AB and O is the centre, then OM $\perp$ AB.]

7. Two circles with centres A and B intersect each other at P and Q and M is the mid-point of PQ (Fig. 12.14). Give reasons for the following statements :

(i) AM ⊥ PQ.

(ii) BM ⊥ PQ

(iii) A, M and B are collinear.

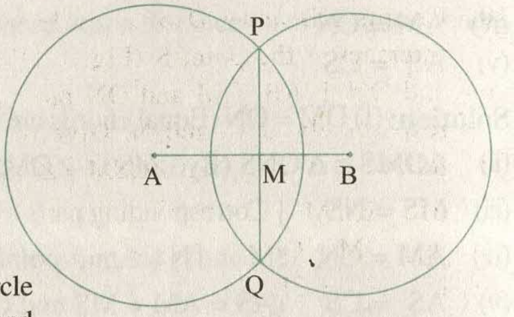


Fig. 12.14

8. AB and BC are two equal chords of a circle with centre O (Fig.12.15). OM ⊥ AB and ON ⊥ BC such that M and N lie on AB and BC, respectively. O and B are joined. Give reasons for the following statements :

(i) OM = ON

(ii) Δ OMB ≅ Δ ONB

(iii) BO bisects ∠ABC.

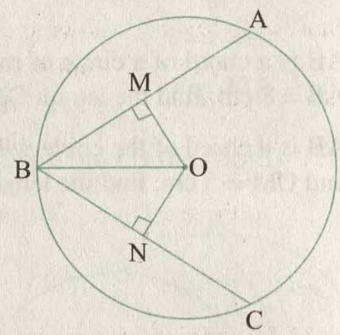


Fig. 12.15

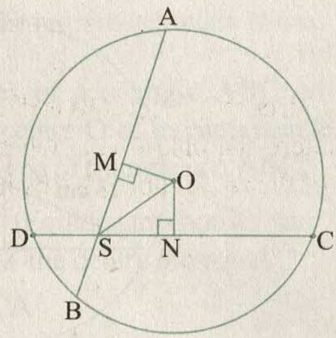


Fig. 12.16

9. AB and CD are two equal chords of a circle with centre O intersecting each other at the point S (Fig.12.16). OM ⊥ AB and ON ⊥ CD such that M and N lie on AB and CD, respectively. OS is joined. Give reasons for the following statements :

(i) OM = ON.

(ii) Δ OMS ≅ Δ ONS.

(iii) MS = NS.

(iv) AS = CS.

(v) BS = DS.

10. Chords AB and CD of a circle with centre O intersect at the point S (Fig. 12.17). OM ⊥ CD and ON ⊥ AB. OM and ON meet AB and CD, respectively at M and N. Give reasons for the following statements, if $\angle OSM = \angle OSN$:

(i) $\Delta OSM \cong \Delta OSN$

(ii) $OM = ON$

(iii) $AB = CD$

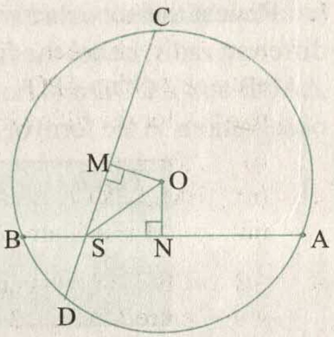


Fig. 12.17

11. State whether the following statements are true (T) or false (F) :

(i) Perpendicular from the centre of a circle to a chord bisects the chord.

(ii) Chords of a circle equidistant from the centre may be unequal.

(iii) In a circle, the line joining the centre to the mid-point of a chord is perpendicular to the chord.

(iv) No three collinear points can lie on a circle.

12.4 Angles Subtended by Equal Chords at the Centre

In Fig. 12.18, AB is a chord of a circle with centre O. $\angle AOB$ is the angle subtended by chord AB at the centre O. We may be interested in knowing whether there exists any relationship between the length of AB and the angle subtended by it at the centre. Let us examine.

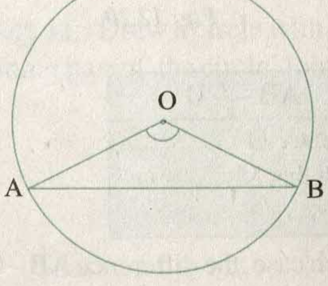


Fig. 12.18

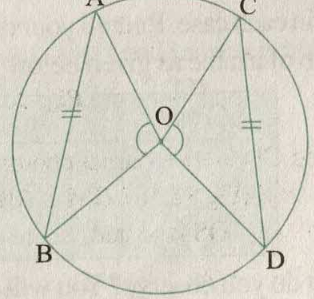


Fig. 12.19

Activity 9 : Draw a circle with centre O and draw two equal chords AB and CD. Join OA, OB, OC and OD (Fig. 12.19).

Repeat the above activity by drawing two more circles with different centres and different radii. Label the figures similarly. Number the circles as 1, 2 and 3. Measure $\angle AOB$ and $\angle COD$ and find the difference $\angle AOB - \angle COD$ in each case. Record your observations in the form of a table as given below :

Circle	$\angle AOB$	$\angle COD$	$\angle AOB - \angle COD$
1.			
2.			
3.			

What do you observe? You will observe that, in each case, $\angle AOB - \angle COD$ is nearly zero. Thus, in all cases, it appears that $\angle AOB = \angle COD$.

Using the SSS property we see that $\Delta OAB \cong \Delta OCD$. This shows that :

Equal chords of a circle subtend equal angles at the centre

Activity 10 : Draw a circle with centre O. Draw four radii OA, OB, OC and OD of this circle in such a way that $\angle AOB = \angle COD$. Join AB and CD (Fig. 12.20).

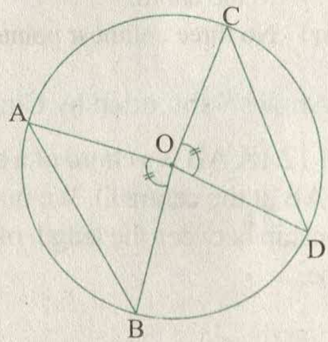


Fig. 12.20

Repeat this activity by drawing two more circles with different centres and different radii. Label the figures similarly. Number the circles as 1, 2 and 3.

Measure AB and CD and find the difference AB – CD in each case. Record your observations in the form of a table as given below :

Circle	AB	CD	AB – CD
1.			
2.			
3.			

What do you observe? You will observe that in each case, the difference AB – CD is nearly zero. Thus, in all cases, it appears that AB = CD.

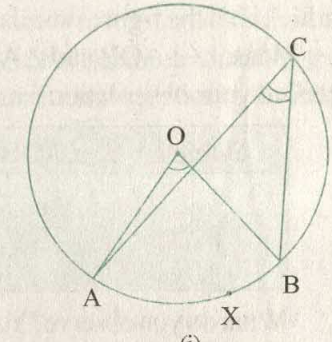
Using the SAS property we see that $\Delta OAB \cong \Delta OCD$. This shows that:

Chords of a circle which subtend equal angles at the centre are equal.

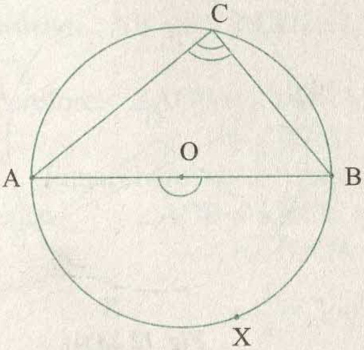
Remark : Note that this proposition is the converse of the previous proposition.

12.5 Angles Subtended by an Arc at the Centre and at any Point on the Remaining Part of the Circle

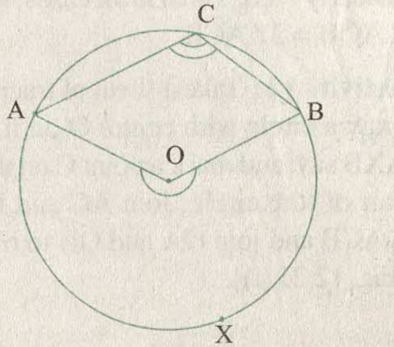
Look at Fig. 12.21 (i), AXB is an arc of a circle with centre O, ∠AOB is the angle subtended by arc AXB at the centre and ∠ACB is the angle subtended by it at any point on the remaining part of the circle. Note that in Fig. 12.21(ii), AXB is a semicircle and the angle AOB subtended by it at the centre is a straight angle, while in Fig. 12.21(iii), AXB is a major arc and the angle AOB subtended by it at O is a reflex angle.



(i)



(ii)



(iii)

Fig. 12.21

We may be interested in knowing whether there exists any relationship between ∠AOB and ∠ACB. Let us examine.

Activity 11 : Draw a circle with centre O. Take an arc AXB and mark a point C on the remaining part of the circle. Join OA, OB, CA and CB (Fig. 12.22).

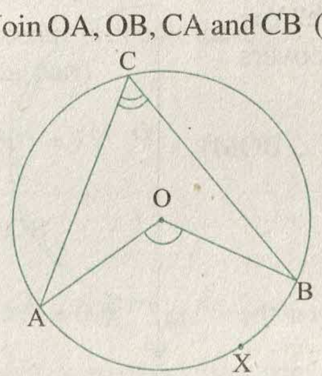


Fig. 12.22

Repeat this activity by drawing two more circles with different centres and different radii. Label the figures similarly. Number the circles as 1,2 and 3.

Measure $\angle AOB$ and $\angle ACB$ and find the difference $\angle AOB - 2\angle ACB$, in each case. Record your observations in the form of a table as given below:

Circle	$\angle AOB$	$\angle ACB$	$2\angle ACB$	$\angle AOB - 2\angle ACB$
1.				
2.				
3.				

What do you observe? You will observe that, in each case, difference $\angle AOB - 2\angle ACB$ is nearly zero. Thus, in all cases, it appears that $\angle AOB = 2\angle ACB$.

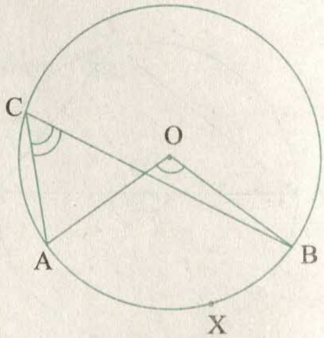


Fig. 12.23 (i)

Activity 12 : Take a sheet of tracing paper and draw a circle with centre O on it. Take an arc AXB say, and mark a point C on the remaining part of the circle. Join AC and BC to obtain $\angle ACB$ and join OA and OB to obtain $\angle AOB$ [Fig. 12.23 (i)].

Now fold the sheet along a line OM passing through O such that A falls on B [Fig. 12.23 (ii)]. In this case, $\angle AOM$ will fall on $\angle BOM$, i.e., $\angle AOM = \angle BOM$. (This means $\angle AOM = \angle BOM = \frac{1}{2}\angle AOB$). Now make a trace copy of $\angle AOM$ (or $\angle BOM$) and place it on $\angle ACB$ after unfolding the circle. What do you observe? You will observe that $\angle AOM$ covers $\angle ACB$ [Fig. 12.23 (iii)].

Thus, $\angle ACB = \angle AOM \; (= \angle BOM)$

i.e., $\angle ACB = \frac{1}{2} \angle AOB$

or $\angle AOB = 2 \angle ACB$

The above two activities suggest the following proposition :

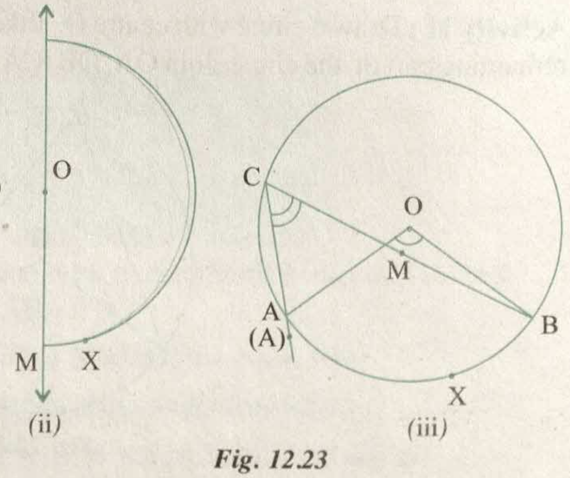


Fig. 12.23

Angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Remarks 1 : $\angle AOB$ subtended by an arc AXB at the centre O of the circle is called the *central angle* of the arc AXB and arc AXB is called the *intercepted arc* corresponding to the central angle AOB.

2. In the case of a semicircle, $\angle ACB = 90°$.

We now take some examples to illustrate these propositions.

Example 4 : In Fig. 12.24, ABCD is a square inscribed in a circle with centre O.
Find $\angle AOB$, $\angle BOC$, $\angle COD$ and $\angle DOA$.

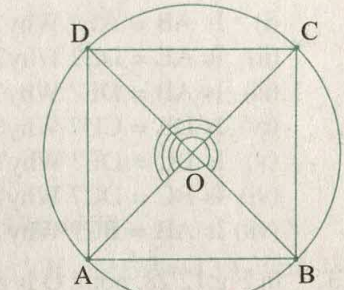


Fig. 12.24

Solution : AB = BC = CD = DA (Sides of a square are equal)

Therefore, $\angle AOB = \angle BOC = \angle COD = \angle DOA$

(Equal chords subtend equal angles at the centre)

Hence, $\angle AOB = \angle BOC = \angle COD = \angle DOA$

$$= \frac{1}{4} \times 360° = 90°$$

Example 5 : Δ ABC is inscribed in a circle with centre O (Fig.12.25). Further, $\angle AOB = 120°$ and $\angle BOC = 150°$.
Find : (i) $\angle BAC$ (ii) $\angle ACB$ (iii) $\angle ABC$

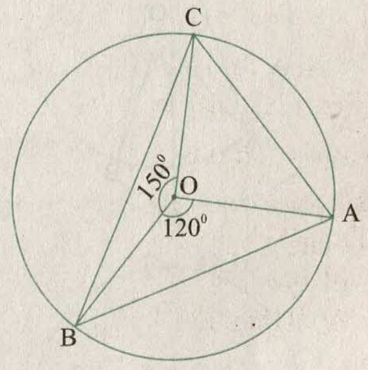


Fig. 12.25

Solution : (i) $\angle BAC = \frac{1}{2} \angle BOC$

(Angle subtended at the centre is double the angle at a point on the remaining part)

$$= \frac{1}{2} \times 150° = 75°$$

(ii) Similarly, $\angle ACB = \frac{1}{2} \angle AOB$

$$= \frac{1}{2} \times 120° = 60°$$

(iii) $\angle ABC = 180° - (75° + 60°)$ (Sum of the three angles of a triangle is 180°)

$= 45°$

EXERCISE 12.2

1. An equilateral triangle ABC is inscribed in a circle with centre O (Fig.12.26). Find $\angle BOC$, $\angle COA$ and $\angle AOB$?

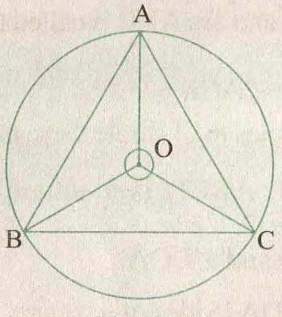


Fig. 12.26

2. In Fig.12.27, a pentagon (five sided closed figure) ABCDE is inscribed in a circle with centre O.

(i) Is AB = AE? Why?
(ii) Is AE = DE? Why?
(iii) Is AB = DE? Why?
(iv) Is DE = CD? Why?
(v) Is BC = DE? Why?
(vi) Is BC = DC? Why?
(vii) Is AB = BC? Why?

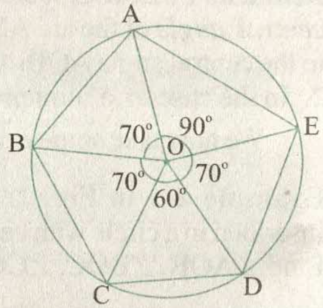


Fig. 12.27

3. In Fig.12.28, point O is the centre of the circle. Find the value of x in each case.

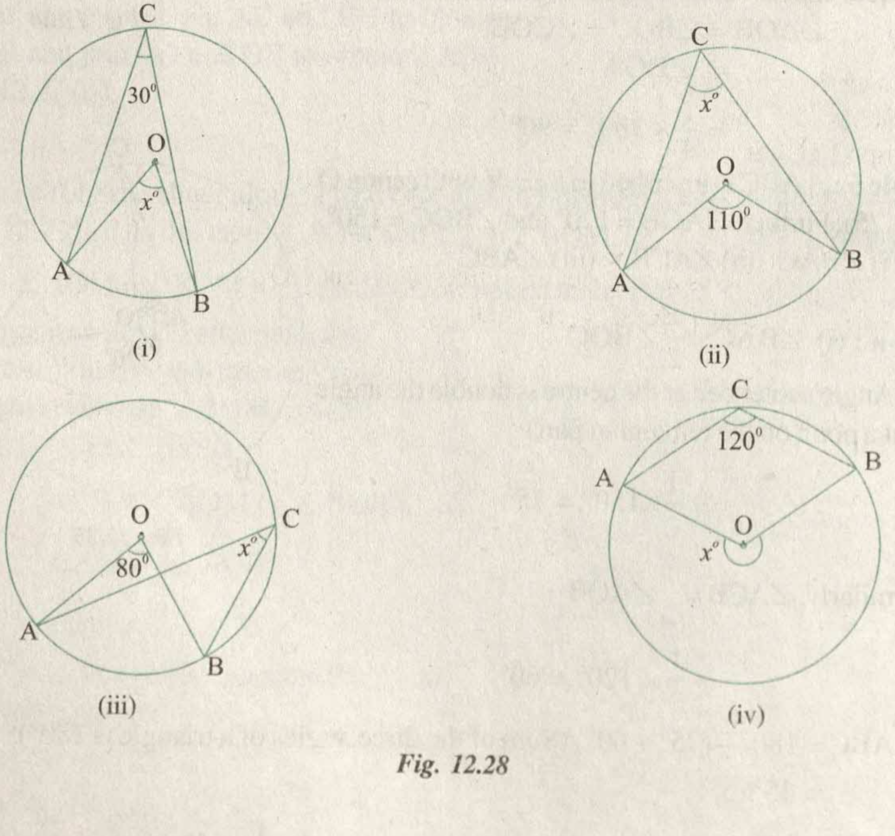


Fig. 12.28

4. In Fig.12.29, O is the centre of the circle and $\angle ABC = 45°$.

(i) Find $\angle AOC$.

(ii) Is $OA \perp OC$?

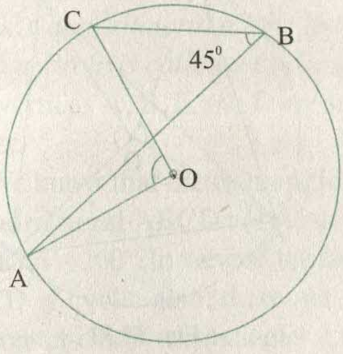


Fig. 12.29

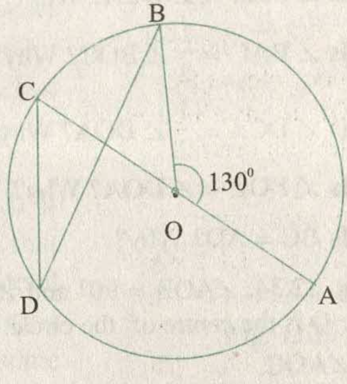


Fig. 12.30

5. In Fig.12.30, AC is a diameter of the circle with centre O. If $\angle AOB = 130°$, find

(i) $\angle BOC$.

(ii) $\angle BDC$.

6. In a circle with centre O, AB is a chord and $OM \perp AB$ and meets the circle at C (Fig.12.31). If $\angle AOC = 80°$, find

(i) $\angle ABC$.

(ii) $\angle MCB$.

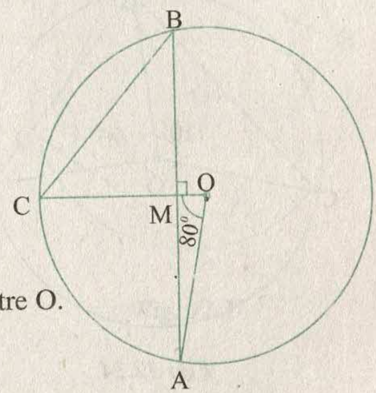


Fig. 12.31

7. In Fig. 12.32, AB is a diameter of a circle with centre O. If $\angle BOC = 30°$ and $\angle COD = 50°$, find

(i) $\angle BAC$.

(ii) $\angle CAD$.

(iii) $\angle AOD$.

(iv) $\angle ABD$.

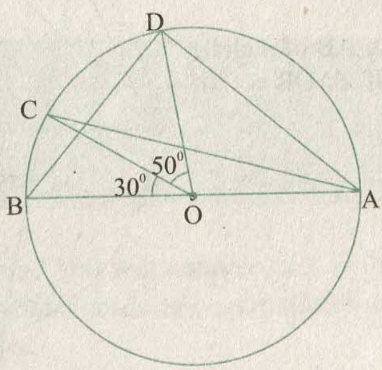


Fig. 12.32

8. ABCD is a trapezium (with AB || DC) inscribed in a circle with centre O (Fig. 12.33). Diagonal AC is joined and also OA, OB, OC and OD are joined.

(i) Is $\angle$ BAC = $\angle$ DCA? Why?

(ii) Is $\angle$ BAC = $\frac{1}{2}$ $\angle$ BOC? Why?

(iii) Is $\angle$ DCA = $\frac{1}{2}$ $\angle$ DOA? Why?

(iv) Is $\angle$ BOC = $\angle$ DOA? Why?

(v) Is BC = AD? Why?

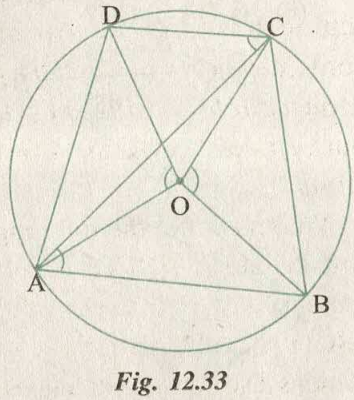


Fig. 12.33

9. In Fig. 12.34, $\angle$AOB = 90° and $\angle$BOC = 110°, where O is the centre of the circle. Find :

(i) $\angle$AOC.

(ii) $\angle$ABC.

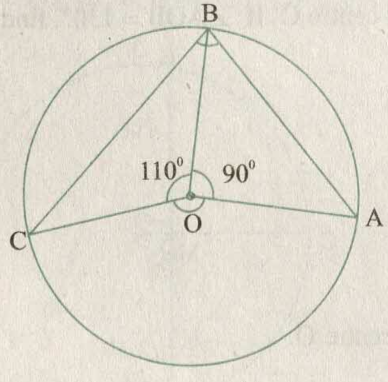


Fig. 12.34

10. P is a point on the minor arc AB of a circle with centre O (Fig. 12.35). If $\angle$AOB = 120° and $\angle$AOP = 75°, find :

(i) $\angle$ARB.

(ii) $\angle$AQP.

(iii) $\angle$ARP.

(iv) $\angle$BRP.

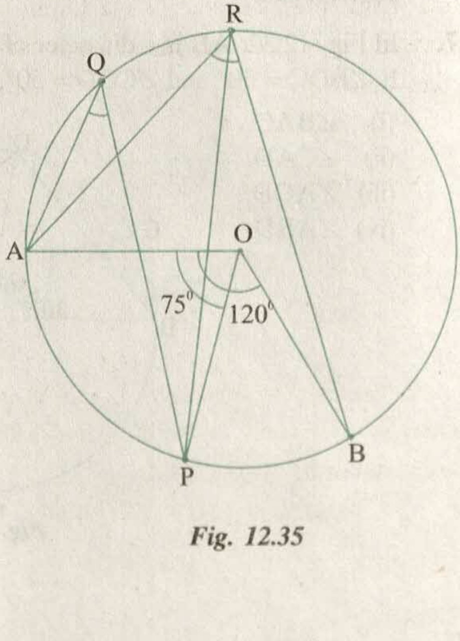


Fig. 12.35

12.6 A Cyclic Quadrilateral and its Angles

In Fig. 12.36, ABCD is a quadrilateral inscribed in a circle. It is called a *cyclic quadrilateral*. In other words, a *quadrilateral all the four vertices of which lie on a circle is called a cyclic quadrilateral*. The four vertices A, B, C and D are said to be *concyclic points*.

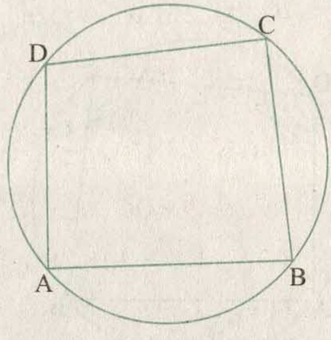


Fig. 12.36

We know that the four angles A, B, C and D of a quadrilateral ABCD are related in the sense that their sum is 360°. In view of the fact that quadrilateral ABCD is cyclic also, there may exist some other relations also between its angles. Let us examine some such relations.

Activity 13 : Draw a circle with centre O and inscribe a quadrilateral ABCD in it (Fig.12.37). Repeat the same activity by drawing two more circles with different centres and different radii. Label the two figures similarly. Number the quadrilaterals as 1, 2 and 3.

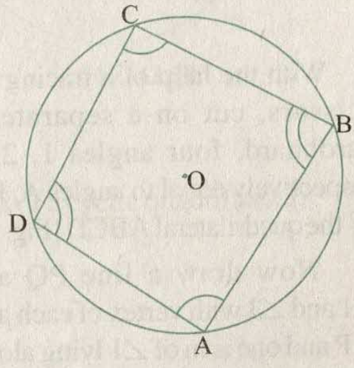


Fig. 12.37

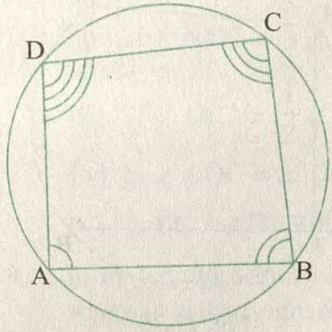
In each case, measure $\angle A$, $\angle B$, $\angle C$ and $\angle D$ and find the sums $\angle A + \angle C$ and $\angle B + \angle D$. Record your observations in the form of a table as given below.

Quadrilateral	$\angle A$	$\angle C$	$\angle A + \angle C$	$\angle B$	$\angle D$	$\angle B + \angle D$
1.						
2.						
3.						

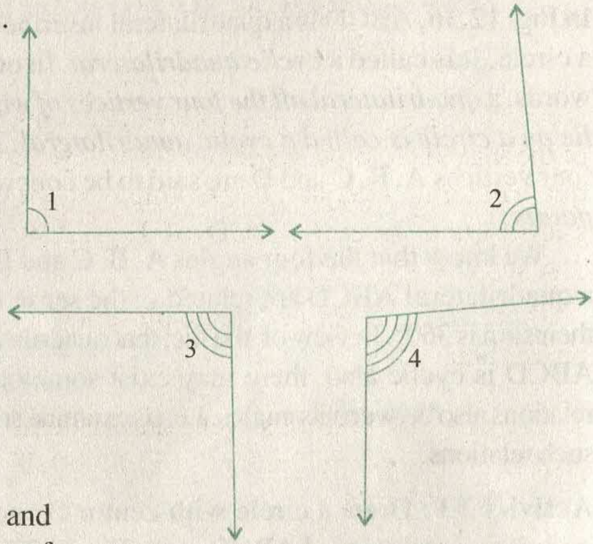
What do you observe? You will observe that, in each case, the sum $\angle A + \angle C$ is nearly equal to 180°. A similar observation is made about the sum $\angle B + \angle D$. Thus, in all cases, it would appear that

$$\angle A + \angle C = 180° \text{ and } \angle B + \angle D = 180°.$$

Activity 14 : Take a cardboard sheet and draw a circle on it. In this circle, inscribe a quadrilateral ABCD [Fig. 12.38 (i)].



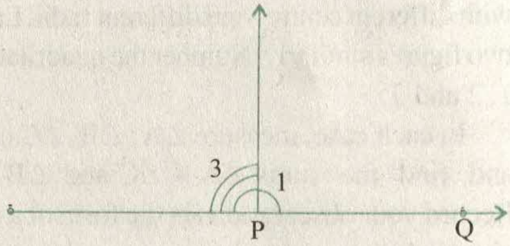
(i)



(ii)

With the help of a tracing paper and scissors, cut on a separate sheet of cardboard, four angles 1, 2, 3 and 4 respectively equal to angles A, B, C and D of the quadrilateral ABCD [Fig. 12.38 (ii)].

Now draw a line PQ and place $\angle 1$ and $\angle 3$ with vertex of each angle lying at P and one arm of $\angle 1$ lying along ray PQ and other arm coinciding with one arm of $\angle 3$ without any overlapping [Fig. 12.38 (iii)]. What do you observe about the other arm of $\angle 3$? You will observe that the other arm of $\angle 3$ lies along the line QP (ray opposite the ray PQ). Thus, $\angle 1 + \angle 3 = 180°$, i.e., $\angle A + \angle C = 180°$

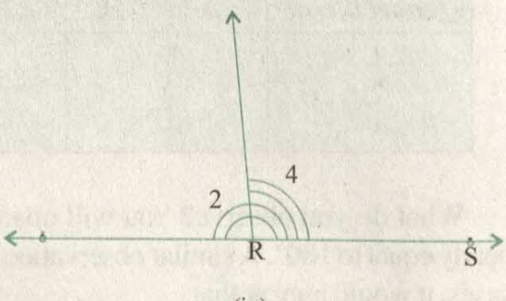


(iii)

Similarly, by taking angles 2 and 4, it can be seen that

$\angle 2 + \angle 4 = 180°$ [Fig.12.38 (iv)],

i.e., $\angle B + \angle D = 180°$



(iv)

Fig. 12.38

The above two activities illustrate the following proposition about opposite angles of a cyclic quadrilateral.

Opposite angles of a cyclic quadrilateral are supplementary.

Remark : From the above, it follows that sums of the angles in pairs of opposite angles of a cyclic quadrilateral are equal. For example, in the cyclic quadrilateral ABCD of Fig.12.39, $\angle A + \angle C = \angle B + \angle D$.

Let us take some examples to illustrate this proposition.

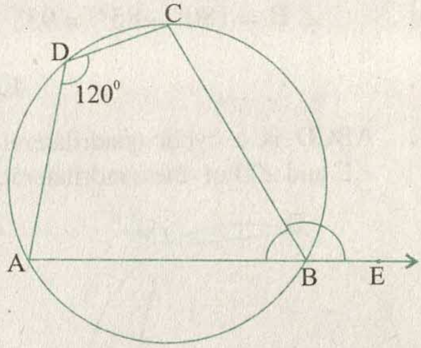


Fig. 12.39

Example 6 : Side AB of a cyclic quadrilateral ABCD is produced to a point E (Fig.12.39). If $\angle ADC = 120°$, find

(i) $\angle ABC$.

(ii) $\angle CBE$.

Solution : (i) $\angle ABC + \angle ADC = 180°$ (Opposite angles of a cyclic quadrilateral)

i.e., $\angle ABC + 120° = 180°$

or $\angle ABC = 60°$

(ii) $\angle ABC + \angle CBE = 180°$ (Linear pair)

i.e., $60° + \angle CBE = 180°$

or $\angle CBE = 180° - 60° = 120°$

Remark : Note that an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Example 7 : In Fig.12.40, $\angle FDE = 85°$ and $\angle C = 70°$. Find (i) $\angle A$, (ii) $\angle D$ and (iii) $\angle B$ of the quadrilateral ABCD.

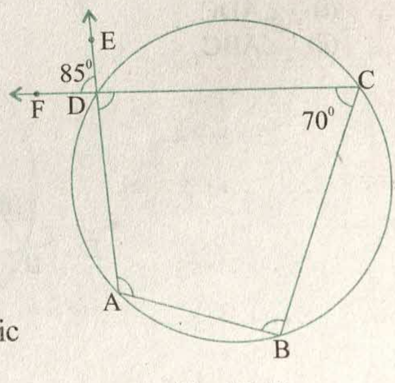


Fig. 12.40

Solution (i) : $\angle C = 70°$ (Given)

Therefore, $\angle A + 70° = 180°$ (Opposite angles of a cyclic quadrilateral)

or $\angle A = 180 - 70° = 110°$

(ii) $\angle ADC = \angle FDE = 85°$ (Vertically opposite angles)

(iii) $\angle B + \angle ADC = 180°$ (Opposite angles of a cyclic quadrilateral)

i.e., $\angle B + 85° = 180°$

or $\angle B = 180° - 85° = 95°$

EXERCISE 12.3

1. ABCD is a cyclic quadrilateral in which $\angle A = 70°$ and $\angle B = 75°$ (Fig. 12.41). Find $\angle C$ and $\angle D$ of the quadrilateral.

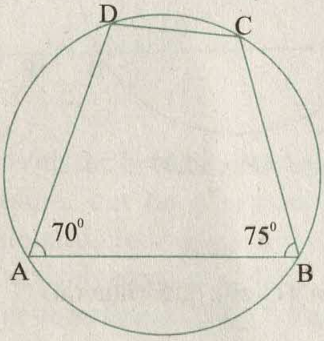


Fig. 12.41

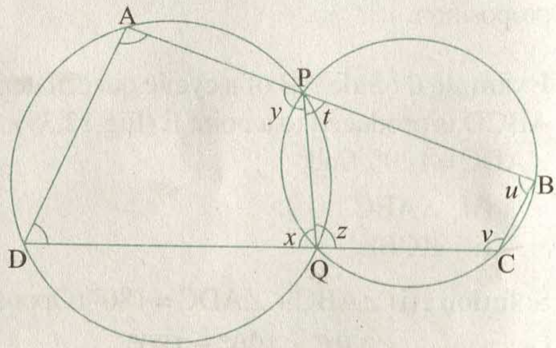


Fig. 12.42

2. Two circles intersect at P and Q. APQD and PQCB are quadrilaterals inscribed in the two circles as shown in Fig.12.42. If $\angle A = 95°$ and $\angle D = 65°$, find the the values of :

(i) x (ii) y (iii) z (iv) t (v) u (vi) v

3. In Fig.12.43, $\angle ADE = 110°$. Find :

(i) $\angle ADC$

(ii) $\angle ABC$

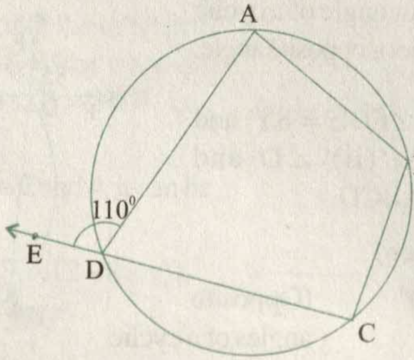


Fig. 12.43

4. In Fig.12.44, $\angle SPR = 70°$, $\angle RSQ = 40°$ and $\angle SRP = 30°$ Find :

(i) $\angle PSQ$

(ii) $\angle PQR$

(iii) $\angle QRS$

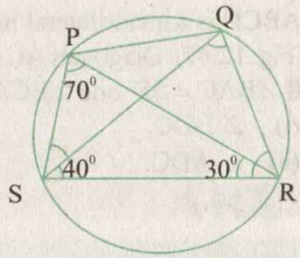


Fig. 12.44

5. ABCD is a cyclic trapezium (trapezium with all the four vertices lying on a circle) with AB || DC (Fig.12.45). Give reasons for the following statements :

(i) $\angle A + \angle D = 180°$.

(ii) $\angle B + \angle D = 180°$.

(iii) $\angle A = \angle B$.

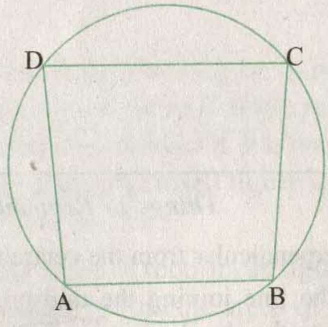


Fig. 12.45

6. ABCD is a parallelogram inscribed in a circle.

(i) Is $\angle A = \angle C$? Why?

(ii) Is $\angle A + \angle C = 180°$? Why?

(iii) Is $\angle A = \angle C = 90°$? Why?

(iv) Is $\angle B = \angle D = 90°$? Why?

(v) Is ABCD a rectangle? Why?

7. ΔABC is inscribed in a circle and P, Q and R are points on the circle as shown in Fig. 12.46. Find :

(i) $\angle P + \angle BAC$

(ii) $\angle Q + \angle ABC$

(iii) $\angle R + \angle ACB$

(iv) $\angle P + \angle Q + \angle R + \angle BAC + \angle ABC + \angle ACB$

(v) $\angle P + \angle Q + \angle R$

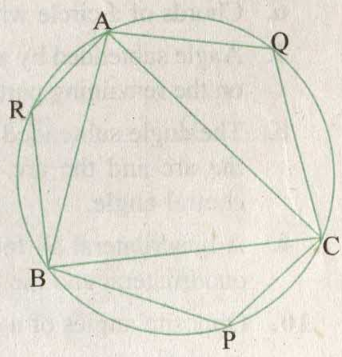


Fig. 12.46

8. ABCD is a quadrilateral inscribed in a circle (Fig. 12.47). Diagonals AC and BD are joined. If $\angle BAC = 55°$ and $\angle BCA = 45°$, find

(i) $\angle$ BDC.

(ii) $\angle$ ADC.

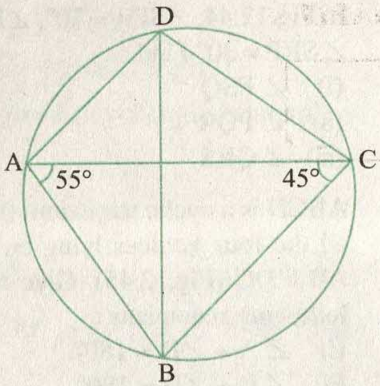


Fig. 12.47

Things to Remember

1. In a circle, perpendicular from the centre to a chord bisects the chord.
2. In a circle, the line joining the mid-point of a chord to the centre is perpendicular to the chord.
3. Equal chords of a circle are equidistant from the centre.
4. Chords equidistant from the centre are equal.
5. Equal chords of a circle subtend equal angles at the centre.
6. Chords of a circle which subtend equal angles at the centre are equal.
7. Angle subtended by an arc at the centre is double the angle subtended by it on the remaining part of the circle.
8. The angle subtended by an arc at the centre is called the central angle of the arc and the arc is called the intercepted arc corresponding to the central angle.
9. A quadrilateral all four of whose vertices lie on a circle is called a cyclic quadrilateral and the four vertices are called concyclic points.
10. Opposite angles of a cyclic quadrilateral are supplementary.

CHAPTER

13

AREAS

13.1 Introduction

Recall that a figure lying in a plane is called a *plane* figure. A plane figure made up of lines or curves or both, is said to be a *closed* curve if it has no free ends. It is called *simple* if it does not cross itself. It is called *rectilinear* if it is made up of line segments only. A part of the plane enclosed by a simple closed figure is called a *plane region.* The magnitude of a plane region is called its *area*.

You already know how to find the areas of rectilinear figures like a square and a rectangle. In this Chapter, we shall learn to find the areas of some more rectilinear figures namely a parallelogram, a triangle and a trapezium. We shall also learn how to find the area of a non-rectilinear figure called circle with which you are familiar. A circle, as you know, is one of the most important human discoveries. It has far reaching consequences in our daily life. We shall find a very interesting property of the circle that connects its diameter and its circumference.

13.2 Parallelogram : A Review

Recall that a parallelogram is a simple closed rectilinear figure consisting of four line segments, called its sides, with opposite sides being parallel. Fig. 13.1 shows a parallelogram ABCD. AB and DC form one pair of opposite sides of this parallelogram and AB || DC. Similarly, AD || BC.

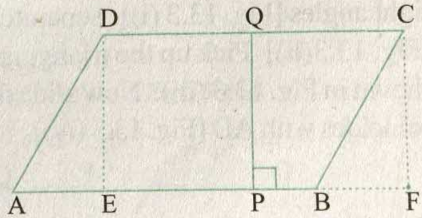


Fig. 13.1

Let us consider the side AB as the base. Generally, the base is drawn horizontally and lower than the other side parallel to it. Consider any line segment with one end point P on the base and one end point Q on the side opposite to the base. If PQ is perpendicular to AB, then PQ is the shortest distance, or simply the distance between

the parallel lines AB and DC. We say that the length *PQ is the height of the parallelogram corresponding to the base AB.* It is usual to show the height by drawing the perpendicular to the base from one of the upper vertices. Thus, when AB is taken as the base, the height may be shown by the line segment DE or CF (Fig.13.1). However, any side of the parallelogram may be taken as its base.

Note that if you take a side from the other pair of parallel sides as the base, then the height of the parallelogram will change. Thus, if BC is taken as the base of the parallelogram in Fig. 13.1, then the height is given by AN or DM (Fig. 13.2).

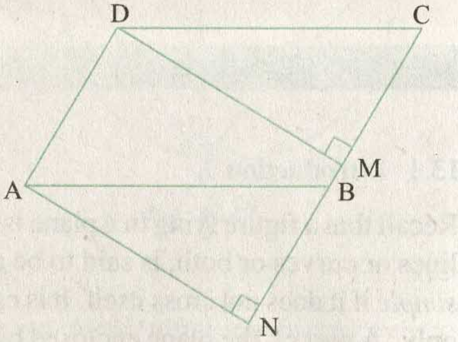


Fig. 13.2

In the rest of the Chapter, we shall use the word *base* for the base of the parallelogram as well as for the length of its base. Also, we shall use the word *altitude* for the line segment showing the height of the parallelogram. Thus, in Fig. 13.1, DE and CF are two altitudes when either AB or DC is considered as the base. As usual, we shall use AB, etc. for the line segment AB as well as its length. We shall generally drop the word *corresponding* in reference to height and base.

13.3 Area of a Parallelogram

We shall now describe a paper cutting activity that will help us in finding the area of a given parallelogram.

Activity 1 : Draw a parallelogram ABCD on a sheet of thick paper. Consider AB as the base. Draw an altitude BF. Then BF $\perp$ DC so that both of the angles DFB and BFC are right angles [Fig. 13.3 (i)]. Separate the triangle BCF by cutting along BC, CF and FB [Fig. 13.3(ii)]. Pick up the triangular piece and place it to the left of the other piece as shown in Fig. 13.3 (iii). Now slide the triangular piece towards the other piece till BC coincides with AD [Fig. 13.3 (iv)].

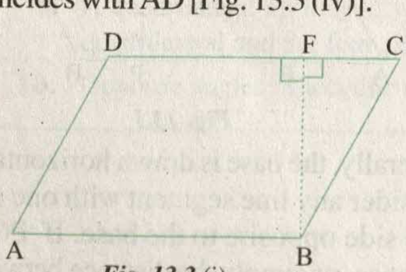


Fig. 13.3 (i)

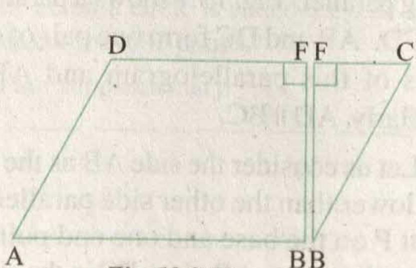


Fig. 13.3 (ii)

The new piece has been obtained from the parallelogram by cutting and pasting without overlaps or gaps. Hence the areas of the two pieces are equal. Now the final piece in

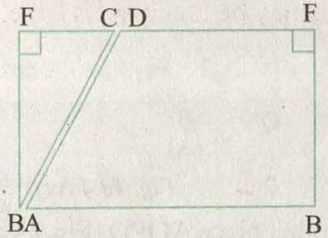


Fig. 13.3 (iii)

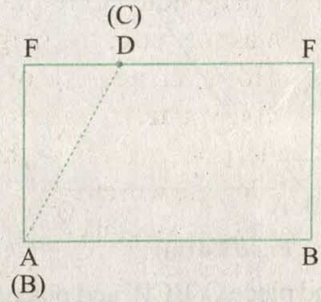


Fig. 13.3 (iv)

Fig.13.3 (iv) is clearly a rectangle with sides AB and BF. Now

$$\begin{aligned}\text{Area of the rectangle} &= \text{Length} \times \text{Breadth}\\ &= AB \times BF\\ &= \text{Base of the parallelogram ABCD} \times \text{its height}\end{aligned}$$

$$\begin{aligned}\text{Hence the area of the parallelogram} &= AB \times BF\\ &= \text{Base} \times \text{Height}\end{aligned}$$

as AB is the base of the given parallelogram and BF is its altitude or height.

Repeat this activity with several other parallelograms and verify that the area of the parallelogram in each case is equal to the product of its base and height.

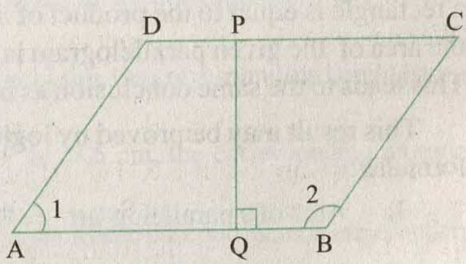


Fig.13.4 (i)

Activity 2 : Draw a parallelogram ABCD. Make a trace copy of the parallelogram and trace it on a stiff paper (or an old greeting card). Take a point P on DC. Draw PQ perpendicular to AB [Fig. 13.4 (i)]. Then PQ is the height of the given parallelogram.

Cut out the parallelogram from the paper/card and also cut along PQ. This divides the given parallelogram into two quadrilaterals AQPD and Q′BCP′ [Fig. 13.4 (ii)]. Take the

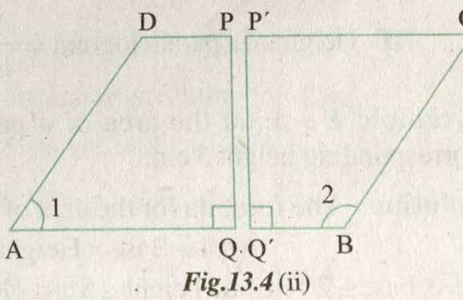


Fig.13.4 (ii)

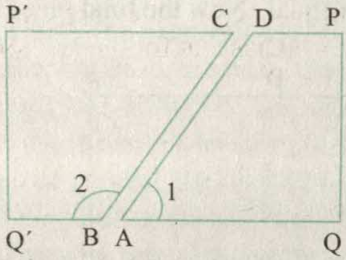


Fig. 13.4 (iii)

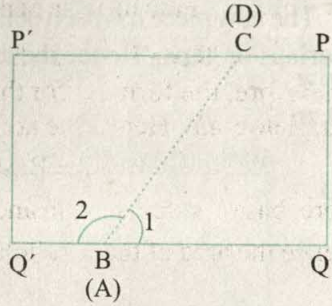


Fig. 13.4 (iv)

right hand piece Q′BCP′ and place it to the left of the other piece AQPD [Fig.13.4 (iii)]. Slide the left piece towards the right piece so that BC coincides with AD [Fig.13.4(iv)].

By the properties of parallel lines, 1 + 2 = 180^0.

Thus, in Fig. 13.4 (iv), 1 and 2 make a linear pair.

Hence Q′BQ (or Q′AQ), i.e. Q′Q is a straight line.

This gives us a rectangle whose base AB and height PQ are respectively equal to the base and height of the given parallelogram.

Since there has been no overlapping, and there have been no gaps, the area of the given parallelogram is the same as that of this rectangle. But we know that the area of a rectangle is equal to the product of its length (base) and breadth (height). Hence, the area of the given parallelogram is equal to the product of its base and its height. This leads to the same conclusion as in Activity 1 above.

This result may be proved by logical arguments also. So we have the following formulas :

I. Area of a parallelogram $=$ Base $\times$ Height,

II. Base of a parallelogram $= \dfrac{\text{Area}}{\text{Height}}$

III. Height of a parallelogram $= \dfrac{\text{Area}}{\text{Base}}$

Example 1 : Find the area of a parallelogram whose base is 20 cm and the corresponding height 5 cm.

Solution : The formula for the area of a parallelogram is

Area $=$ Base $\times$ Height

Here, base $= 20$ cm and height $= 5$ cm. Hence the required area $= 20 \times 5\ \text{cm}^2 = 100\ \text{cm}^2$.

Example 2 : Find the area of a rhombus whose side is 6.5 cm and whose altitude is 4 cm.

Solution : Recall that a rhombus is a parallelogram, whose all sides are equal. Therefore, the formula for the area of a rhombus is the same as that for the area of a parallelogram. Hence the area of the rhombus is given by

$$\text{Area} = \text{Base} \times \text{Height}$$

Here, base = side = 6.5 cm and height = length of the altitude = 4 cm.

Hence the area of the parallelogram = $6.5 \times 4 \text{ cm}^2 = 26 \text{ cm}^2$.

Example 3 : Find the base of a parallelogram whose area is 400 cm^2 and whose height is 8 cm.

Solution : We know that

$$\text{Base} = \frac{\text{Area}}{\text{Height}}$$

Here, area = 400 cm^2 and height = 8 cm.

Hence base = $\frac{400}{8}$ cm = 50 cm

EXERCISE 13.1

1. Find the area of a parallelogram whose base is 12 cm, the corresponding height being 7 cm.
2. Find the area of a parallelogram whose base is 12 dm, the corresponding height being 5 dm.
3. Find the area of a parallelogram whose base is 28.5 cm, the corresponding altitude being 10 cm.
4. Find the area, in square metres, of the parallelogram whose base and altitude are as under:
   (i) base = 12 dm, altitude = 100 dm
   (ii) base = 124 cm, altitude = 10 dm
   (iii) base = 9 m, altitude = 90 cm
   (iv) base = 15 cm, altitude = 9 cm
5. Find the area of a rhombus whose side is 6 cm and whose altitude is 4 cm.
6. Find the area of a rhombus whose side is 6.5 cm and whose altitude is 40 dm.
7. Find the altitude of a parallelogram one of whose sides is 6.5 cm, the area being 26 cm^2.
8. Find the altitude of a parallelogram one of whose sides is 10 cm, the area being 0.5 m^2.

9. Find the base of a parallelogram whose area is 390 cm^2 and height is 26 cm.

10. Find the base of a parallelogram whose area is 560 m^2 and altitude is 1400 cm.

11. Find the altitude of a rhombus whose area is 420 cm^2 and perimeter is 140 cm.

12. Two sides of a parallelogram are 20 cm and 25 cm. If the altitude corresponding to the sides of length 25 cm is 10 cm, find the altitude corresponding to the other pair of sides. [*Hint :* Find the area first.]

13. The base and the corresponding altitude of a parallelogram are 10 cm and 12 cm, respectively. If the other altitude is 8 cm, find the length of the other pair of parallel sides.

14. A floral design on the floor of a building consists of 2800 tiles. Each tile is in the shape of a parallelogram of altitude 3 cm and base 5 cm. Find the cost of polishing the design at the rate of 50 paise per dm^2.

13.4 Area of a Triangle

We shall now describe a paper cutting activity that will help us in finding the area of a given triangle.

Activity 3 : Draw a triangle ABC. Let AL be the altitude corresponding to the base BC (Fig.13.5). Through A and C, draw lines parallel to BC and BA respectively, meeting in the point D.

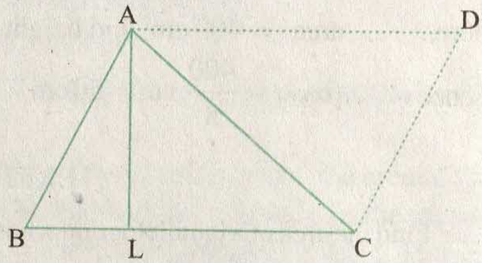


Fig. 13.5

Since BA || CD and AD || BC, therefore, ABCD is a parallelogram. Also, AL is an altitutde corresponding to the base BC. Hence

$$\text{Area of parallelogram ABCD} = BC \times AL \qquad (1)$$

Now cut along the line segments AC, CD and DA. This removes Δ CDA from the parallelogram ABCD. The parallelogram now reduces to Δ ABC with which we started.

Place ΔCDA over Δ ABC in such a manner that C falls on A, A falls on C, and D is on the same side of AC as B. You will find that D actually falls on B. This is not surprising in view of the following facts :

1. Alternate angles BAC and ACD are equal (BA || CD and AC cuts them.). This means CD falls along AB.

2. AB = DC. This means D falls on B.

It becomes clear now that CDA exactly covers . This means that the areas of the two triangles are equal. Since the parallelogram consists of these two triangles only, we have

$$\text{Area of parallelogram ABCD} = \text{Area of } \Delta\,\text{CDA} + \text{Area of } \Delta\,\text{ABC}$$

$$= 2 \times (\text{Area of } \Delta\,\text{ABC})$$

$$(\text{Area of } \Delta\,\text{ABC} = \text{Area of } \Delta\,\text{CDA})$$

or $$\text{Area of } \Delta\,\text{ABC} = \frac{1}{2}(\text{Area of parallelogram ABCD})$$

$$= \frac{1}{2}(\text{BC} \times \text{AL}) \qquad [\text{From (1)}]$$

$$= \frac{1}{2}(b \times h),$$ where b is the base and h the height or altitude of Δ ABC

Hence, we have the following formulas :

I. $\text{Area of a triangle} = \frac{1}{2}(\text{Base}) \times (\text{Altitude}) = \frac{1}{2}(\text{Base}) \times (\text{Height})$

II. $\text{Base of a triangle} = \dfrac{2 \times \text{Area}}{\text{Height}} = \dfrac{2 \times \text{Area}}{\text{Altitude}}$

III. $\text{Height (Altitude) of a triangle} = \dfrac{2 \times \text{Area}}{\text{Base}}$

Remark : Observe that the area of a triangle is half the area of a parallelogram on the same base and height.

Example 4 : Find the area of a triangle whose base is 24 cm and height is 14 cm.

Solution : Area A of a triangle is given by

$$A = \frac{1}{2}(b \times h).$$

Here, $b = 24$ cm and $h = 14$ cm. Therefore,

$$A = \frac{1}{2}(24 \times 14)\,\text{cm}^2 = 168\ \text{cm}^2$$

Example 5 : Find the height of a triangle whose base is 80 cm and area is 0.08 m^2.

Solution : We first convert the area into cm^2 as the base is given in cm. (We could have converted the base into m but then we would have to work with decimals.) Now,

$$1 \text{ m}^2 = 10000 \text{ cm}^2$$

Therefore, $\quad 0.08 \text{ m}^2 = 0.08 \times 10000 \text{ cm}^2 = 800 \text{ cm}^2$

Using Formula III above,

$$\text{Height of the triangle} = \frac{2 \times \text{Area}}{\text{Base}} = \frac{2 \times 800}{80} \text{ cm} = 20 \text{ cm}$$

Example 6 : Find the area of a rhombus, the lengths of whose diagonals are 80 cm and 60 cm.

Solution : Recall that a rhombus is a parallelogram and its diagonals bisect each other at right angles. Let us call the vertices of the rhombus A, B, C and D. Call the point of intersection of the diagonals O (Fig. 13.6). Then we may think of the rhombus as consisting of the four right triangles OBC, OCD, ODA and OAB.

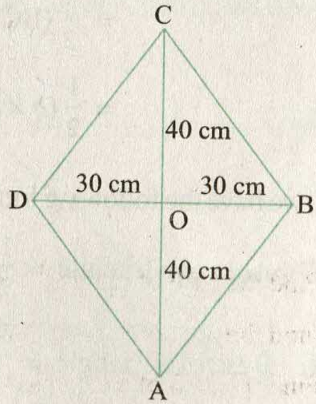


Fig. 13.6

Observe that if we take one leg of the right angle in the triangle OBC as the base, then the other leg is the altitude. Therefore, the area of this triangle is

$$\frac{1}{2} \times \text{OB} \times \text{OC} = \frac{1}{2} \times 30 \times 40 \text{ cm}^2 = 600 \text{ cm}^2$$

Note that the other three triangles are equal in area to this triangle. Hence the area of the given rhombus is

$$4 \times 600 \text{ cm}^2 \text{ i.e., } 2400 \text{ cm}^2$$

Example 7 : Find the area of an equilateral triangle of sides 20 cm each.

Solution : Consider an equilateral triangle ABC such that AB = BC = CA = 20 (in cm). Draw CD AB, meeting AB in D (Fig. 13.7). Then D is the mid-point of AB.

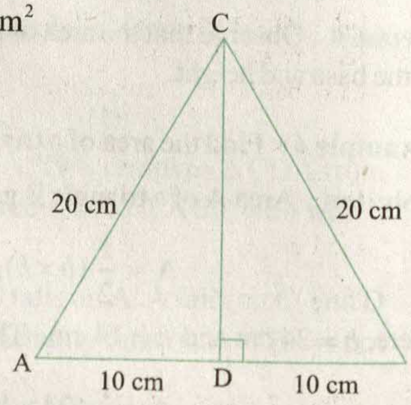


Fig. 13.7

$\therefore \quad AD = DB = 10$ (in cm)

Since DBC is a right triangle, therefore, by Pythagoras Theorem,

$$BC^2 = DB^2 + DC^2$$

or $$20^2 = 10^2 + DC^2$$

or $$DC^2 = 400 - 100$$

or $$DC = \sqrt{300} = \sqrt{3 \times 100} = 10\sqrt{3}$$

Now, area of $\Delta\, ABC = \frac{1}{2}\text{Base} \times \text{Height}$

$$= \frac{1}{2} \times 20 \times 10\sqrt{3}\ \text{cm}^2$$

$$= 100\sqrt{3}\text{cm}^2$$

EXERCISE 13.2

1. Find the area of a triangle with base 18 cm and the corresponding height 7 cm.
2. Find the area of a triangle whose base is 120 dm, height being 75 dm.
3. Find the area of a triangle whose base is 24 cm and whose altitude is 1.5 dm.
4. Find the area, in square metres, of the triangle whose base and altitude is as under :
   (i) base = 12 dm, altitude = 10 dm
   (ii) base = 62 cm, altitude = 50 cm
   (iii) base = 8 m, altitude = 80 cm
   (iv) base = 1500 cm, altitude = 90 dm
5. Find the height of a triangle whose base is 60 cm and whose area is 600 cm^2.
6. Find the height of a triangle whose area is 65 cm^2 and whose base is 13 cm.
7. Find the altitude of a triangle whose base is 6.5 cm and area is 26 cm^2.
8. Find the base of a triangle whose altitude is 10 cm and area is 0.5 m^2.
9. Find the base of a triangle whose area is 3.9 m^2 and whose height is 260 cm.
10. Find the area of an equilateral triangle of sides 30 cm each.
11. Find the area of an equilateral triangle of sides 8 dm each.

12. Find the area of an isosceles right triangle of equal sides 40 cm each.

13. Find the area of a tile in the shape of a rhombus whose diagonals have lengths :
(i) 24 cm and 10 cm (ii) 50 cm and 100 cm (iii) 20 cm and 28 cm

14. A field is in the form of a right triangle with hypotenuse 50 m and one side 40 m. Find the area of the field.

15. A field is in the form of a triangle. If its area is 2 ha and the length of its base is 200 m, then find its altitude.
[*Hint :* 1 ha = 10000 m^2]

16. The length of one of the diagonals of a field in the form of a quadrilateral is 42 m. The perpendicular distances of the other two vertices from this diagonal are12 m and 9 m (Fig. 13.8). Find the area of the field.

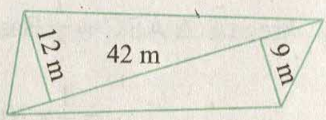


Fig. 13.8

17. Area of a Δ ABC is P square units.
(i) D is the mid-point of AB. Find the areas of triangles ADC and DBC, and show that the two are equal.
[*Hint:* If h is the height of Δ ABC corresponding to the base AB, then

$$P = \frac{1}{2} h \times AB.]$$

(ii) Points E and F divide AB into three equal parts. Find the areas of triangles AEC, EFC and FBC, and show that the three are equal.
(iii) State how to divide Δ ABC into n triangles with equal areas.

13.5 Area of a Trapezium

Recall that a trapezium is a quadrilateral that has two of its sides parallel to each other. Fig.13.9 shows a trapezium ABCD. The sides AB and DC are parallel. Any one of these parallel sides may be taken as the base of the trapezium. The distance between these sides (bases) is the height or the altitude of the trapezium. In Fig.13.9, CL and AM are two altitudes.

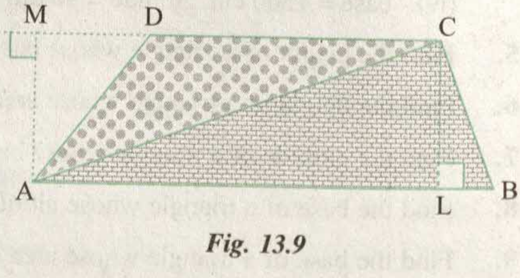


Fig. 13.9

Let us draw the diagonal AC. This divides the trapezium into the two triangles ABC and ACD. Let us denote the two bases AB and DC by b_1 and b_2 respectively. Let us denote the height by h.

Area of trapezium ABCD = Area of Δ ABC + Area of Δ ACD

$$= \frac{1}{2} AB \times CL + \frac{1}{2} DC \times AM$$

$$= \frac{1}{2} b_1 \times h + \frac{1}{2} b_2 \times h$$

$$= \frac{1}{2} h \times (b_1 + b_2)$$

Hence, the area of a trapezium is half the product of the height and the sum of the two bases. Denoting the area by A, we have the following formula :

Area of a trapezium A = $\frac{1}{2} h \times (b_1 + b_2)$, where h is the height and b_1, b_2 are the two bases.

Example 8 : The parallel sides of a trapezium are 20 m and 15 m long. The distance between these sides is 10 m. Find the area of the trapezium.

Solution : The area of a trapezium is given by

$$A = \frac{1}{2} h \times (b_1 + b_2).$$

Here, b_1 = 20 m, b_2 = 15 m and h = 10 m.

$\therefore$ $A = \frac{1}{2} 10 \times (20 + 15)\ m^2 = 175\ m^2$

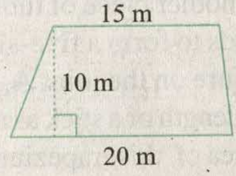


Fig. 13.10

Example 9 : The area of a trapezium of height 7 cm is 140 cm². If one of the parallel sides is 25 cm, find the other side.

Solution : We know that $A = \frac{1}{2} h \times (b_1 + b_2)$.

Here, A = 140 cm², h = 7cm and b_1 = 25 cm. Substituting these values in the above formula,

$140\ cm^2 = \frac{1}{2} \times 7\ cm \times (25\ cm + b_2)$

or $40\ cm = 25\ cm + b_2$

or $b_2 = 15\ cm$

Hence, the length of other side is 15 cm.

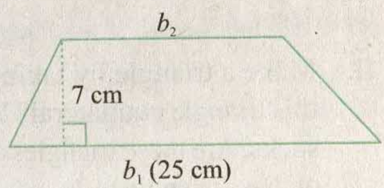


Fig. 13.11

Do it with your friends 1: I . Take a match box. Take out some matchsticks and remove the phosphoric material from the ends. Buy a metre length of cycle valve tube from a cycle repair shop. Cut it into small pieces of length 1.5 cm or so. Now take a piece of tubing and two match sticks. Push one stick into the tubing from one end and the other from the other end so that the sticks touch each other inside the tubing (Fig. 13.12). This is how you join two sticks. Now take five sticks and four pieces of tubing. Join these as shown in Fig. 13.13.

Fig. 13.12

Fig. 13.13

Take another piece of tubing. Push the ends A and B above into the tubing from the two ends to form a five-sided figure (a *pentagon*) [Fig. 13.14 (i)]. Now apply a little pressure on the joint A-B to turn the figure into a trapezium [Fig. 13.14 (ii)]. Define the length of a stick as a unit. (You may give this unit any fancy name you like.) Find the area of this trapezium in terms of the unit defined by you.

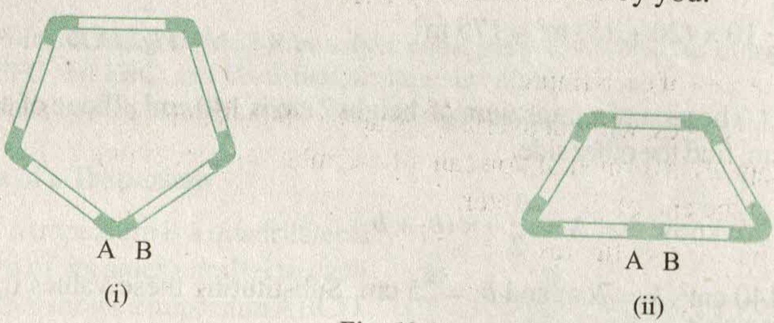


Fig. 13.14

II. Make a triangle by taking 3 sticks (Fig.13.15). Is this triangle equilateral? Make triangles with 6 or 9 sticks. Are these triangles equilateral ? Find the areas of these triangles.

Fig. 13.15

Do it with your friends 2 : Two trapezia, two right triangles and one parallelogram are shown in Fig. 13.16. Make trace copies of these figures.

With the help of the trace copies, cut out the pieces from a piece of cardboard (or an old greeting card).

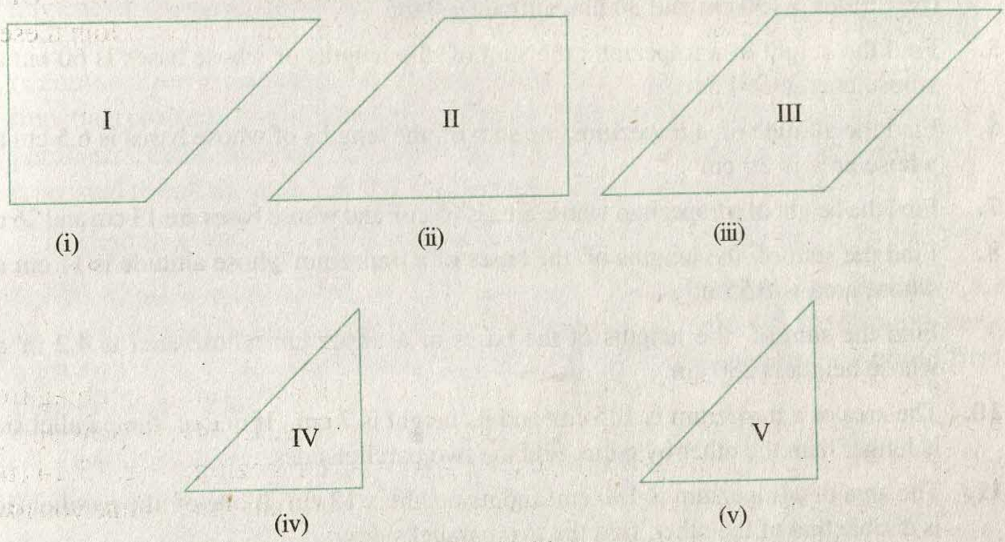


Fig. 13.16

I. Find the area of each figure.

II. Place the pieces, without overlapping and without leaving gaps, so that the letter 'T' (Capital) is formed. If you cannot assemble the pieces into a 'T', look at Fig. 13.28 at the end of the Chapter.

III. Try to make other figures from the pieces.

EXERCISE 13.3

1. Find the area of a trapezium with base 15 cm and height 8 cm if the side parallel to the given base is 9 cm long.

2. Find the area of a trapezium whose parallel sides are of length 16 dm and 22 dm and whose height is 12 dm.

3. Find the area of a trapezium whose bases are 24 cm and 16.4 cm and whose altitude is 1.5 dm.

4. Find the area, in square metres, of the trapezium whose bases and altitude are as under :

(i) bases = 12 dm and 20 dm, altitude = 10 dm

(ii) bases = 28 cm and 3 dm, altitude = 25 cm

(iii) bases = 8m and 60 dm, altitude = 40 dm

(iv) bases = 150 cm and 30 dm, altitude = 9 dm

5. Find the height of a trapezium, the sum of the lengths of whose bases is 60 cm and whose area is 600 cm^2.

6. Find the altitude of a trapezium, the sum of the lengths of whose bases is 6.5 cm and whose area is 26 cm^2.

7. Find the height of a trapezium whose area is 65 cm^2 and whose bases are 13 cm and 26 cm.

8. Find the sum of the lengths of the bases of a trapezium whose altitude is 11 cm and whose area is 0.55 m^2.

9. Find the sum of the lengths of the bases of a trapezium whose area is 4.2 m^2 and whose height is 280 cm.

10. The area of a trapezium is 105 cm^2 and its height is 7 cm. If one of the parallel sides is longer than the other by 6 cm, find the two parallel sides.

11. The area of a trapezium is 180 cm^2 and its height is 12 cm. If one of the parallel sides is double that of the other, find the two parallel sides.

12. In Fig.13.17, AB || DC and DA is perpendicular to AB. Further, DC = 7 cm, CB = 10 cm and AB = 13 cm. Find the area of the quadrilateral ABCD.

[*Hint :* Draw the perpendicular from C to AB meeting it in M. Find CM from the triangle CMB.]

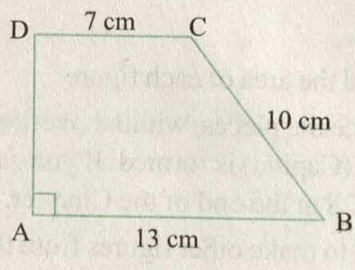


Fig. 13.17

13. The parallel sides DC and AB of a trapezium are 12 cm and 36 cm respectively (Fig.13.18). Its non-parallel sides are each equal to 15 cm. Find the area of the trapezium.

[*Hint :* Through C, draw a line parallel to DA meeting AB in M. Find the altitude of the isosceles triangle CMB.]

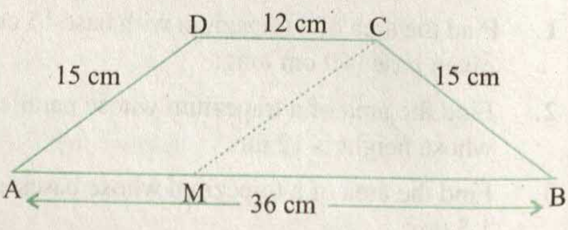


Fig. 13.18

13.6 Circumference of a Circle

A silversmith has to make 100 silver bangles of a given size. For this purpose, he has to buy the required length of silver wire. In order to decide how much wire is needed, he has to know the circular length of a bangle. You know how to measure straight lengths by using any one of the standard measures like cm, dm, m or km. But have you any clue as to how to measure the length around a circle as in case of the bangle above? A circular shape occurs so often in daily life that the problem of measuring the length around a circle is an important problem. As you know, we call the lengths around rectilinear figures their respective perimeters. In the same way, we could call the length around a circle its perimeter. However, it is usual to call the perimeter of a circle by a special name *circumference.*

The length around a circle or the perimeter of a circle is known as its circumference.

We shall now try to find out how to measure the circumference of a circle. Later, we shall obtain a formula that gives us an approximate value of the circumference of a given circle. If we could find a way to find the length of arcs, we could find the circumference also.

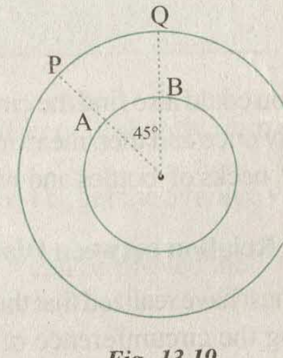


Fig. 13.19

You know that as the angle subtended by a circular arc at the centre of the circle increases, the length of the arc also increases. You may feel that this fact may help us in finding the circumference of a circle. But look at Fig. 13.19. It shows two circles having the same centre. The arc AB of the smaller circle subtends an angle of 45^0 at the centre. The arc PQ of the bigger circle also subtends an angle of 45^0 at the centre. But obviously, the two arcs are not equal. Thus, simply knowing the angle at the centre cannot help us in measuring the arc, and hence, the circumference of a circle.

To get a feeling of what can really help us in measuring the circumference, we shall carry out some measuring activities.

Activity 4 : Take a circle. As remarked earlier, we cannot measure it like a straight length. Make a trace copy of the circle on a thick sheet of paper, or on a piece of cardboard. Cut along the circle so that you have a disc. Mark a point P on the rim of the disc. Note that the rim is really the circle, and the length around the rim is the circumference of this circle.

Now draw a line on a sheet of paper. Mark a point Q on it [Fig.13.20 (i)]. Hold the disc vertically and place it over the line in such a manner that point P lies on point Q

[Fig.13.20 (ii)]. Gently roll the disc along the line, without slipping, in the clockwise direction [Fig. 13.20 (iii)]. Go on rolling until point P falls on the line again.[Fig.13.20 (iv)].

Since there has been no slipping, the distance around the rim of the disc is the same as the distance QP or PQ. Hence, the length of QP or PQ is the circumference of the given circle.

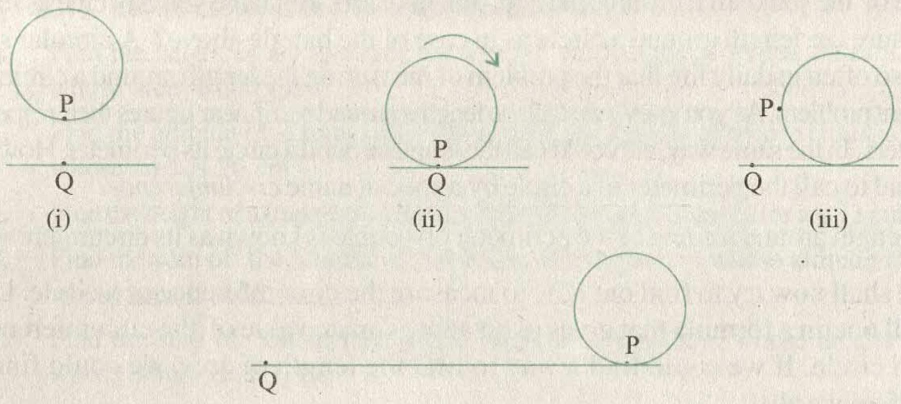


Fig. 13.20

You could also find the circumference by winding a thread around the rim of the disc exactly once and then measuring the length of the thread used. Bottle caps, cylindrical boxes, necks of bottles and wooden discs may be convenient for this purpose.

13.7 Relation between Diameter and Circumference

You must have realized that the above method of finding the circumference of a circle is rather tedious. We shall now carry out another activity that will help us in evolving a formula to calculate the circumference directly by its radius or diameter. Given a circle with centre known, a chord through the centre (with extremities on the circle), gives a diameter of the circle. If the centre is not known, you could find the diameter as the largest chord. You could perhaps slide a ruler across the circle and go on noting the lengths of the various chords [Fig. 13.21 (i)]. A more precise method would be to make a trace copy of the circle and

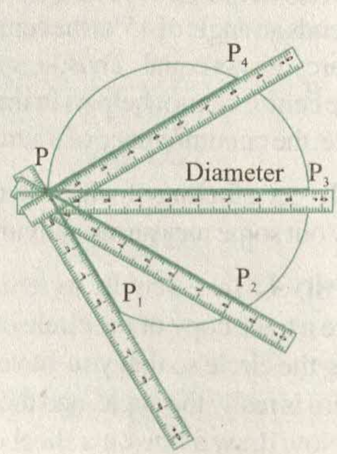


Fig. 13.21 (i)

fold it over itself to produce a semi-circle. The straight distance between the two ends of this semi circle is the diameter [Fig. .13.21 (ii)].

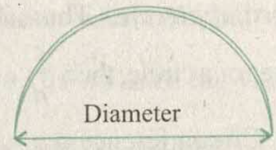


Fig. 13.21 *(ii)*

Activity 5: Take four discs of different sizes. Name these discs P, Q, R and S. Measure the diameter (d) and the circumference (c) of each disc as described above. Calculate the ratio $\frac{c}{d}$ correct to two decimal places, and fill it in the table below :

Disc	*Circumference* (c)	*Diameter* (d)	*Ratio* $\left(\frac{c}{d}\right)$
P			
Q			
R			
S			

You will find that the values of $\frac{c}{d}$ in the last column differ only slightly from each other, if at all. Now add all the four values of $\frac{c}{d}$. Divide by 4 to get the average value of $\frac{c}{d}$. The average value is likely to be near 3.1. In fact, it can be proved that:

I. The ratio $\left(\frac{c}{d}\right)$ of the circumference (c) of a circle to its diameter(d) is the same, that is, constant for all circles.

II. The constant ratio of the circumference of a circle to its diameter is written as π. It has the approximate value 3.14 correct to two decimal places.

13.8 The Number π

We have just observed that the circumference of a circle is in a constant ratio to its diameter. We have denoted this constant ratio by the Greek letter π (pi). We read it as *pie* in

the word *apple-pie*. Thus, if c and d denote respectively the circumference and the diameter of a circle, then $\frac{c}{d} = \pi$. Thus, we have the following relations :

$$\text{Circumference} = \pi \times \text{diameter}, \text{ i.e., } c = \pi d \qquad \text{(I)}$$

$$\text{Circumference} = \pi \times 2 \times \text{radius} = 2\pi \times \text{radius, i.e., } c = 2\pi r. \qquad \text{(II)}$$

From I and II, we also get the relations :

$$d = \frac{c}{\pi}, \text{ and } r = \frac{c}{2\pi} \qquad \text{(III)}$$

Recall that 3.14 is an approximate value of π. You might feel that it will be a better idea to find the exact value of π instead of using the value correct to two decimal places. Unfortunately, it does not work. You may calculate the value of π upto trillions of decimal places, but it would still be an approximate value. Not surprising! Mathematicians have proved that π is not a rational number. Therefore, it cannot be written as a terminating (finite) or a non-terminating repeating decimal.

The above fact about π has given rise to some interesting activities. All the world over, people have tried to beat others in calculating the value of π to more and more places. In September 1999, Dr. Kanada of the University of Tokyo calculated 206,158,430,000 decimal digits of π. In September 2002, he and his team broke their own world record, calculating 1.2411 trillion digits (over six times more than before) of π. This activity may appear to be useless to some people, but calculating π to as many digits as possible in a given period of time using the same method is used as a test to compare the power of different computers. One hundred and eighty decimal digits (digits after the decimal point) in the value of π are given below :

1415926535	8979323846	2643383279	5028841971
6939937510	5820974944	5923078164	0628620899
8628034825	3421170679	8214808651	3282306647
0938446095	5058223172	5359408128	4811174502
8410270193	8521105559		

From here, the value of π correct to 3, 4, 5 and 6 places of decimals are respectively :

3.142, 3.1416, 3.14159 and 3.141593

Early mathematicians used different approximations for the value of π. Babylonians used a very rough estimate 3, for the value of π. Early Greeks used the value $\frac{22}{7}$. Archimedes (around 250 B.C.) proved that the value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. The Indian mathematician Aryabhata (476 – 550) gave the value of π as $\frac{62832}{20000}$ which comes out correct to four decimal places. Earlier values were all less accurate.

For ease of computations, we shall use the value $\frac{22}{7}$ *of* π *unless stated otherwise.*

Example 10 : Find the circumference of a circle whose diameter is 35 cm.

Solution : We know that $c = \pi d$.

Here, $d = 35$ cm. Therefore, taking $\pi = \frac{22}{7}$,

$$c = \frac{22}{7} \times 35 \text{ cm} = 110 \text{ cm}$$

Hence, the circumference of the circle is 110 cm.

Example 11: Find the circumference of a cycle wheel whose radius is 3.7 dm. (Use $\pi = 3.14$.)

Solution : We know that $c = 2\pi r$.

Here, $r = 3.7$ dm. Taking $\pi = 3.14$,

$$c = 2 \times 3.14 \times 3.7 \text{ dm} = 23.236 \text{ dm}$$

Hence, the circumference of the cycle wheel is 23.236 dm.

Example 12 : Find the diameter of a circular puddle whose circumference is 220 cm.

Solution : We know that $d = \frac{c}{\pi}$. Here, $c = 220$ cm. Taking $\pi = \frac{22}{7}$,

$$d = \frac{220}{\frac{22}{7}} \text{ cm} = \frac{220 \times 7}{22} \text{ cm} = 70 \text{ cm}$$

Hence, the diameter of the puddle is 70 cm.

Example 13 : We have a rope just sufficient to encircle a circular region of radius 100 m. Show that with a rope only 7 m longer than the given rope, we can encircle a circular region of radius 101 m.

Solution : Let us first find the rope required to encircle the circle of radius 100 m. The formula $c = 2\pi r$ gives the length of the required rope as $2\pi \times 100$ m. This means we have $200\,\pi$ m long rope with us.

Similarly, the rope required to encircle the circle of radius 101 m is $2\pi \times 101$ m or $202\,\pi$ m.

The difference in the lengths is

$$202\,\pi\text{ m} - 200\,\pi\text{ m} = 2\,\pi\text{ m or } \frac{44}{7}\text{ m which is less than 7 m.}$$

Thus, with a 7 m longer rope than the previous rope, we can encircle the bigger circle.

EXERCISE 13.4

Unless specified, use $\pi = 3.14$ *only if the values are given as decimal number.*

1. Find the circumference of a circle, whose diameter is
   (i) 14 cm (ii) 11 dm (iii) 20 m
2. Find the circumference of a circle whose radius is
   (i) 2.5 cm (ii) 1.50 dm (iii) 0.25 m
3. Find the diameter of a circle whose circumference is
   (i) 12.56 cm (ii) 88 dm (iii) 15.70 m
4. Find the radius of a circle whose circumference is
   (i) 6.28 cm (ii) 2200 dm (iii) 308 m
5. The diameter of a coin is 2 cm. Find its circumference.
6. The circumference of a dinner plate is 75.36 cm. Find its radius.
7. Two circles having the same centre have radii 350 m and 490 m. What is the difference in their circumferences ?
8. The diameter of a cycle wheel is 70 cm. Find how many times the wheel will revolve in order to cover a distance of 110 m.
9. A water sprinkler in a lawn sprays water as far as 7 m in all directions. Find the length of the outer edge of wet grass.

10. An ox in a *kolhu* (an oil pressing apparatus) is tethered to a rope 3 m long. How much distance does it cover in 14 rounds?

11. A circular piece of thin wire is converted into a square of side 6.25 cm. If there is no loss or gain in its length, find the radius of the circular wire.

12. The ratio of the radii of two wheels is 3 : 4. What is the ratio of their circumferences?

13. A piece of thin wire in the form of an equilateral triangle of side 31.4 dm is bent into a ring with no loss in wire. Find the diameter of the ring.

14. A well of diameter 150 cm has a stone parapet around it. If the length of the outer edge of the parapet is 660 cm, then find the width of the parapet.

15. A circular pond has a 90 cm wide footpath along its edge. A man walks around the outer edge of the footpath with 66 cm long steps. In 400 steps, he makes a full round. What is the radius of the pond?
[*Hint :* Radius of the pond = Radius of the outer edge of the path – Width of the path.]

16. The moon is about 384000 km from the earth and its path around the earth is nearly circular. Find the circumference of the path described by the moon in one complete revolution about the earth. [Use $\pi = 3.14$.]

17. Consider the number 113355 formed by the first three odd numbers 1, 3 and 5. Form another number $\frac{355}{113}$ by putting the first three digits (113) of this number in the denominator and the remaining three digits in the numerator. Express this number as a decimal. What does this decimal representation have to do with π ?

13.9 Area of a Circle

As you know, the circle is not a rectilinear figure. However, it is plane, closed, simple and bounded. So it encloses a plane region. The magnitude of the region enclosed by a circle is called the area of the circle. We shall now carry out an activity that will help us in finding a formula for the area of a given circle.

Activity 4 : Draw a circle with radius r on a tracing paper. Cut along the circle to get a circular disc. Fold one half of the disc over itself [Fig. 13.22 (i)] so that each part coincides with the other. Press along the diameter (that forms the straight side of the semi circular figure so obtained) to get a crease. Fold again to get a quarter circle [Fig. 13.22 (ii)]. Press along the bounding radii to get creases. Open out the disc. Cut along the creases to

separate out the four quarters [Fig. 13.22 (iii)]. Place these four quarters as shown in Fig. 13.22 (iv). The area of the circle is the total area covered by this figure.

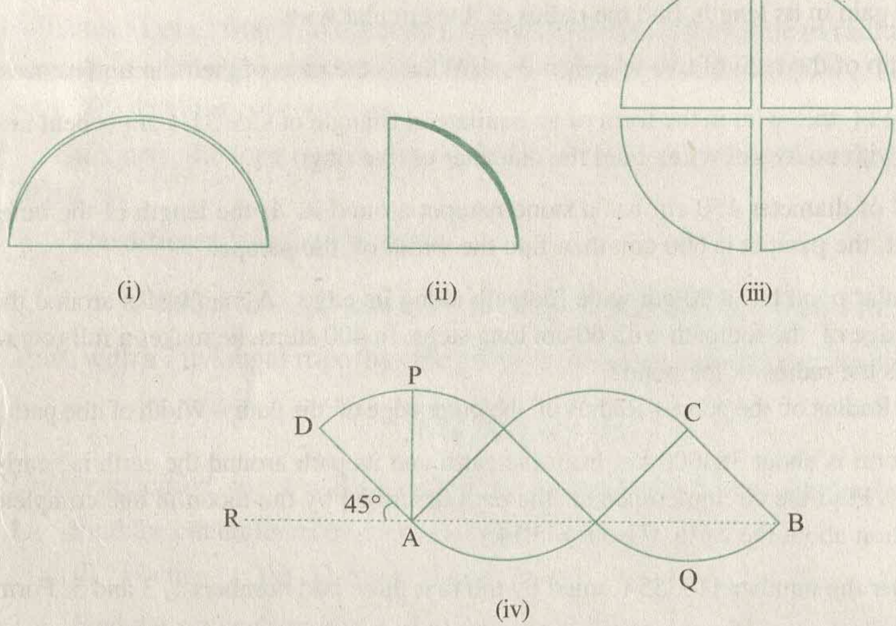


Fig. 13.22

Note that :

(i) AD and BC are line segments each equal to the radius of the given circle.

(ii) The length of the curvilinear part between A and B is equal to half of the circumference, and therefore, equal to πr. Similarly, the curvilinear part between D and C is also equal to πr. The curvilinear length between P and D is half of a quarter circumference. Therefore, it is $\frac{1}{8}$ th of the circumference.

(iii) $\angle PAD = 45^\circ$ and $\angle RAD = 45^\circ$.

Again start with a circle of the same radius and fold it into a quarter as before. Fold the quarter to get an eighth part of the circle. Press to get a crease [Fig.13.23 (i)]. Open out the disc. Cut along the creases to separate out the eight parts.

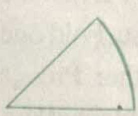


Fig. 13.23 (i)

Place these eight parts as shown in Fig. 13.23 (ii). The area of the circle is the total area covered by this figure.

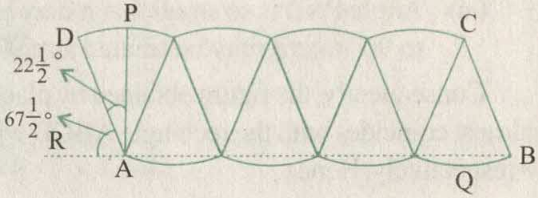


Fig. 13.23 (ii)

Note that :

(i) AD and BC are line segments each equal to the radius of the given circle.

(ii) The length of the curvilinear part between A and B is equal to half of the circumference, and therefore, equal to πr. Similarly, the curvilinear part between D and C is also equal to πr. The curvilinear length between P and D has reduced to $\frac{1}{16}$ th of the circumference. Thus, D is closer to P now. Similarly B is closer to Q in this case. This means that the line segments DC and AB are coming closer to the curvilinear length (πr) between them.

(iii) $\angle PAD = 22\frac{1}{2}°$ and $\angle RAD = 67\frac{1}{2}°$. Thus, $\angle PAD$ is decreasing and $\angle RAD$ is increasing.

Again start with a circle of the same radius and fold it into 16th parts. Press to get a crease. Cut out the 16 parts and place these 16 parts as shown in Fig. 13.24. The area of the circle is the total area covered by this figure.

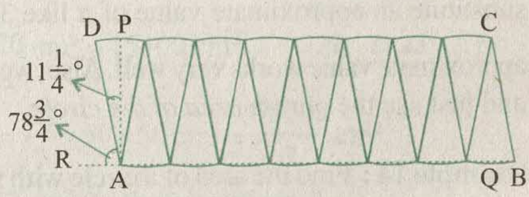


Fig. 13.24

Note that :

(i) AD and BC are line segments each equal to the radius of the given circle.

(ii) D is closer to P now than in the previous case. Similarly, B is closer to Q than before. This means lengths DC and AB are coming closer to πr.

(iii) $\angle PAD = 11\frac{1}{4}°$ and $\angle RAD = 78\frac{3}{4}°$. Thus, $\angle PAD$ is decreasing and $\angle RAD$ is increasing.

We may repeat the above procedure to divide the circle into more and more parts and place the parts side by side to get a figure such that :

(i) Two of the sides (AD and BC) are equal to the radius of the circle.

(ii) AB and DC differ very little from πr.

(iii) Angle PAD is so small that it may be treated as zero. Angle $\angle$RAD is so close to 90° that it may be treated as a 90° angle.

Consequently, the figure obtained by placing the parts of the given circle side by side almost coincides with the rectangle ABCD whose length and breadth are equal to πr and r respectively. Hence,

$$\begin{aligned}\text{Area of the given circle} &= \text{Total area of the parts}\\ &= \text{Area of rectangle ABCD}\\ &= \pi r \times r = \pi r^2\end{aligned}$$

We have the following formula for the area A of a circle with radius r :

$$A = \pi r^2 \text{ or area of a circle} = \pi\,(\text{radius})^2$$

From here, we also have

$$r = \sqrt{\frac{A}{\pi}} \text{ or radius} = \sqrt{\frac{\text{Area}}{\pi}}$$

Remark : When we use the above formulae to calculate the area A or the radius r, we substitute an approximate value of π like 3.14 or $\frac{22}{7}$. For all practical purposes, an approximate value works very well. Also, we shall generally omit the word approximate and just use the phrase *area of the circle*.

Example 14 : Find the area of a circle with radius equal to 1.20 cm. [Take $\pi = 3.14$.]

Solution : We know that

Area of a circle $= \pi r^2$.

Here, $r = 1.20$ cm. Taking $\pi = 3.14$,

$$\begin{aligned}\text{Area of the circle} &= 3.14 \times (1.20)^2 \text{ cm}^2\\ &= 4.52 \text{ cm}^2 \text{ correct to 2 decimal places}\end{aligned}$$

Thus, the required area of the circle is 4.52 cm^2.

Example 15 : Find the radius of a circle whose area is equal to 5544 cm^2.

Solution : We know that radius of a circle $= \sqrt{\frac{\text{Area}}{\pi}}$.

Here, area = 5544 cm^2. Taking $\pi = \frac{22}{7}$,

$$\text{Radius of the circle (in cm)} = \sqrt{\frac{5544}{\frac{22}{7}}} = \sqrt{\frac{5544 \times 7}{22}} = \sqrt{252 \times 7}$$

$$= \sqrt{6 \times 6 \times 7 \times 7} = 42$$

Thus, the radius of the circle is 42 cm.

Example 16 : A circular rug of diameter 1.4 m is spotted in the centre with a 20 cm striped border (Fig 13.25). Find the area of the striped portion of the rug to the nearest cm^2.

Solution : Radius of the rug = Half of the diameter of the rug

$$= \frac{1}{2} \times 1.4 \text{ m} = 0.7 \text{ m}$$

$$= 70 \text{ cm}$$

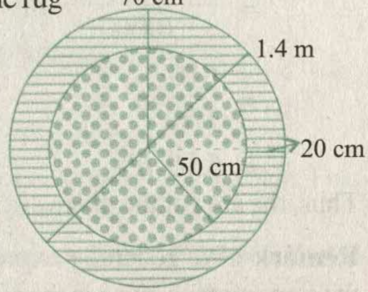


Fig. 13.25

Radius of the spotted portion

= Radius of the rug – Width of the striped border

= 70 cm – 20 cm = 50 cm

Now, area of the rug $= \pi r^2 = \frac{22}{7} \times 70 \times 70 \text{ cm}^2 = 15400 \text{ cm}^2$,

area of the spotted portion $= \pi r^2 = \frac{22}{7} \times 50 \times 50 \text{ cm}^2 = \frac{55000}{7} \text{ cm}^2$

$= 7857 \text{ cm}^2$ (to the nearest cm^2)

Hence, the area of the striped border (to the nearest cm^2)

= Area of the rug – Area of the spotted portion

$= 15400 \text{ cm}^2 - 7857 \text{ cm}^2$

$= 7543 \text{ cm}^2$

Example 17 : Find the area of a circle whose circumference is 880 cm.

Solution : We know that the circumference c is given by the relation $c = 2\pi r$, where r is the radius.

Here, $c = 880$ cm.

so that $880 = 2\pi r = 2 \times \frac{22}{7} \times r$ $\left[\text{Taking } \pi = \frac{22}{7}\right]$

or $\quad r = \dfrac{880 \times 7}{2 \times 22} = 140$

Hence, the radius of the given circle is 140 cm.

Now, area A of the circle with radius r is given by

$$
\begin{aligned}
A &= \pi r^2 \\
&= \frac{22}{7} \times 140^2 \text{ cm}^2 \\
&= 22 \times 140 \times 20 \text{ cm}^2 \\
&= \frac{22 \times 140 \times 20}{10000} \text{ m}^2 \quad [\text{As } 1\text{m}^2 = 10000 \text{ cm}^2] \\
&= \frac{616}{100} \text{ m}^2 \\
&= 6.16 \text{ m}^2
\end{aligned}
$$

Thus, the area of the given circle is 6.16 m^2.

Remark : We generally express the answer in the same units as given in the problem. However, if the figures are large, then the answer may be converted into higher units.

Do it with your friends 3 : Draw any right triangle ABC, right angled at A. Draw semi-circles on the sides AB, BC and CA (Fig. 13.26).

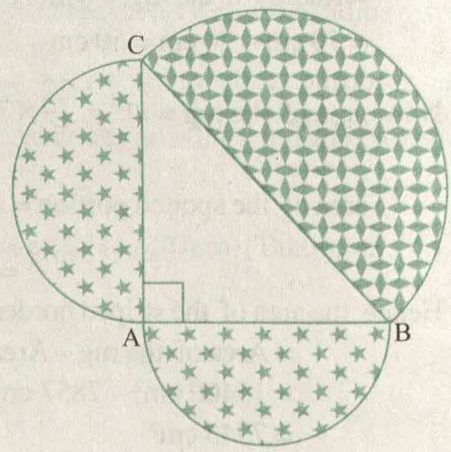


Fig. 13.26

(i) Find the sum of the areas of the semi-circle on the legs AB and AC.

(ii) Find the area of the semicircle on the hypotenuse.

(iii) What relation do you find in (i) and (ii)? Repeat this activity with other right triangles.

(iv) State your observation like Pythagoras Theorem.

(v) Repeat the activities (i) to (iv) above, taking equilateral triangles in place of semicircles.

EXERCISE 13.5

1. Find the area of a circle whose radius is :

   (i) 21 cm (ii) 49 dm (iii) 217 cm

2. Find the area of a circle whose diameter is :

   (i) 20 cm (ii) 9.8 dm (iii) 200 cm

3. Find the radius of a circle whose area is :

   (i) 154 cm^2 (ii) 616 dm^2 (iii) 12474 cm^2

4. Find the radius of the circle whose area is :

   (i) 1386 cm^2 (ii) $\frac{2200}{7}$ dm^2 (iii) $\frac{13750}{7}$ cm^2

5. Using the value 3.14 of π, find the diameter of the circle whose area is :

   (i) 314 cm^2 (ii) 7850 dm^2 (iii) 4710 cm^2

6. Find the area of a circular plate with diameter 10 cm.

7. An ox is tied to a pole with a 10 m long rope. The ox moves keeping the rope tight. Find the area of the ground swept by the rope.

8. A crater falls near a village creating a circular pit of diameter 200 m. Find the affected area of land.

9. A horse is tied to a pole fixed at one corner of a 30 m × 30 m square field of grass, by means of a 10 m long rope (Fig 13.27). [Take $\pi = 3.14$.]

   (i) Find the area of that part of the field in which the horse can graze.

   (ii) Find the increase in the grazing area if the rope were 20 m long instead of being 10 m long.

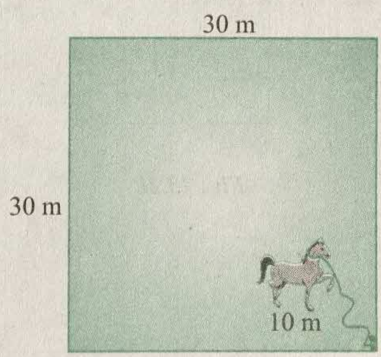


Fig. 13.27

10. What will be the area in part (i) of the above question if the pole were fixed on a side, somewhere near the middle ?

11. Find the area of a circle whose circumference is the same as the perimeter of a square of side 11 m.

12. Which has greater area, a square of perimeter 88 cm or a circle with circumference 88 cm?

13. From a rectangular metal sheet of sides 30 cm and 40 cm, a circular sheet as big as possible, is cut off. Find the area of the remaining sheet.

14. A well of diameter 150 cm has a 30 cm wide parapet running around it. Find the area of the parapet.

15. The areas of two circles are in the ratio 25 : 36. Find the ratio of their circumferences.

16. The radius of a circle is doubled. What is the ratio of the area of the new circle to the area of the given circle?

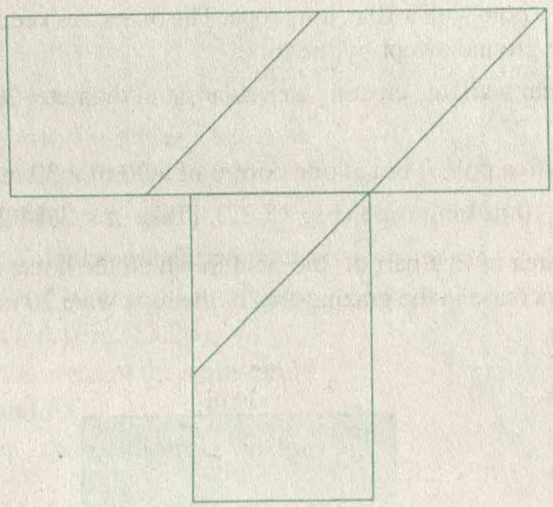


Fig. 13.28

Things to Remember

1. The magnitude of a plane region is called its area.
2. Area of a parallelogram = Base × Altitude (height) or $A = b \times h$.
3. Area of a triangle = $\frac{1}{2}$ Base × Altitude (height) or $\mathrm{A} \quad \frac{1}{2}b \quad h$.
4. Area of a trapezium = $\frac{1}{2}$ (Sum of bases) × Altitude (height) or

   $\mathrm{A} = \frac{1}{2}\left(b_1 + b_2\right) \times h$.
5. The perimeter of a circle is called its circumference.
6. The ratio $\frac{c}{d}$ of the circumference (c) of a circle to its diameter (d) is a constant number for all the circles.
7. The constant ratio $\frac{c}{d}$ of the circumference of a circle to its diameter is denoted

   by the Greek letter π. Thus, $\frac{c}{d} = \pi$. The approximate value of π is 3.14 correct to two decimal places.
8. The number π is not a rational number. An often-used rational approximation to π is $\frac{22}{7}$.
9. Circumference of a circle = 2π × (radius) or $c = 2\pi r$.
10. Circumference of a circle = π × (diameter) or $c = \pi d$.
11. Area of a circle = π × (radius)^2 or $\mathrm{A} = \pi r^2$.
12. Radius of a circle = $\sqrt{\text{Area} \div \pi}$ or $r = \sqrt{\frac{\mathrm{A}}{\pi}}$.

CHAPTER

14

SURFACE AREAS

14.1 Introduction

You are already familiar with the concept of surface area from your earlier classes. In Class VII, you have learnt about the surface areas of two simple three dimensional figures (solid figures), viz. cuboids and cubes. Recall that the surface area of a cuboid of length l, breadth b and height h units is $2(lb + bh + hl)$ square units and that of a cube of length (edge) l units is $6l^2$ square units. In this Chapter, we shall be concerned with surface areas of three familiar solid figures – the cylinder, the cone and the sphere. As the surface of these solids is generally curved, it creates certain difficulties when we want to find the surface areas of these solids. We shall get over these difficulties by finding an equivalent plane region which enables us to find the area of these curved surfaces.

The formulas for surface areas of these solids are very useful because we come across cylinders, cones and spheres in our everyday life almost at every step. These formulas involve the use of number π. In this Chapter also, we shall take the value of π as $\frac{22}{7}$ unless mentioned otherwise.

14.2 Right Circular Cylinder

A tin can, a road roller (or a garden roller), round pillars, cables, water pipes, etc. (Fig. 14.1) are some of the objects from daily life which suggest or bring to our mind the concept of a right circular cylinder, which is a geometric figure.

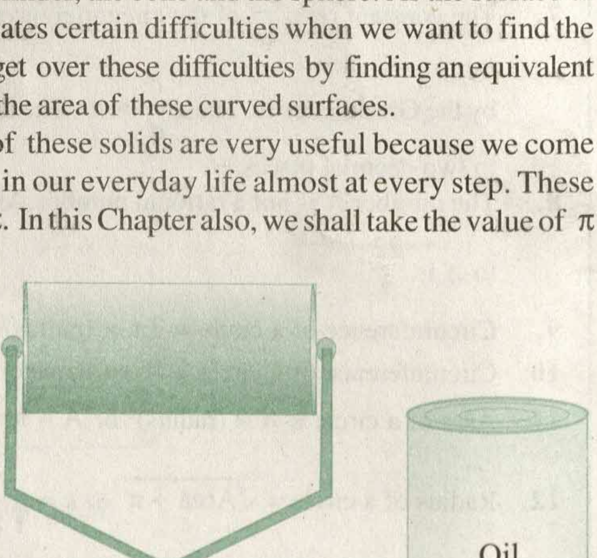


Fig. 14.1

Outline of a right circular cylinder is given in Fig. 14.2. It will help us to describe a right circular cylinder in geometric terms already known to us as follows :

A right circular cylinder has two plane ends. Each plane end is circular in shape, i.e., each end is a circular region. These two circular regions are congruent and parallel to each other. Each of these ends is called a *base* of the cylinder. The line segment OO′ joining the centres of the two plane ends is called the *axis* of the cylinder. Note that OO′ is perpendicular to every line segment lying in each of these plane ends passing through O or O′. In other words, it is *perpendicular* to these circular ends (bases). For this reason, we name this solid figure as a *Right Circular Cylinder*.

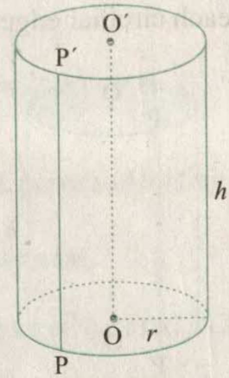


Fig. 14.2

There is a *curved* (not flat) surface which joins the two ends and we call it the *lateral surface* of the right circular cylinder. Observe that for each point P on the circle of the lower end (base), there is a point P′ on the circle of the upper end such that PP′ is parallel to OO′. As P goes around the lower circle, the line segment PP′ generates (or produces) the entire lateral (or curved) surface of the cylinder. The radius *r* of the circle (base) and the length *h* of the line segment OO′ are the two lengths that determine the size of the cylinder. *h* is called the *height* of the cylinder. Note that PP′ = OO′.

Remark : The above description of a right circular cylinder brings to our mind two distinct but related figures : the *hollow cylinder* and the *solid cylinder*. In fact, by a right circular cylinder we mean a hollow right circular cylinder. It is the figure formed in space by just the lateral surface of the cylinder. The part of the space enclosed by a right circular cylinder is called its *interior*. A right circular cylinder along with its interior is called a *right circular cylindrical region*, which is generally referred to as a *solid right circular cylinder*. In common usage, the word right circular cylinder is used for both, the hollow right circular cylinder and the solid right circular cylinder. In practice, it should not cause any problem, for the context in which it is used, will make the meaning clear.

Further, unless stated otherwise, we shall often use the word 'cylinder' to mean 'right circular cylinder'.

14.3 Surface Area of a Right Circular Cylinder

Let us now try to determine the surface area of a cylinder. For this, we perform the following activity :

Activity 1: Let us consider a right circular cylinder of height h. Let the radius of the base be r [Fig. 14.3 (i)]. Thus, each of the ends of the cylinder is a circle of radius r. Note that each circular edge has length $2\pi r$ and area of each plane end is πr^2.

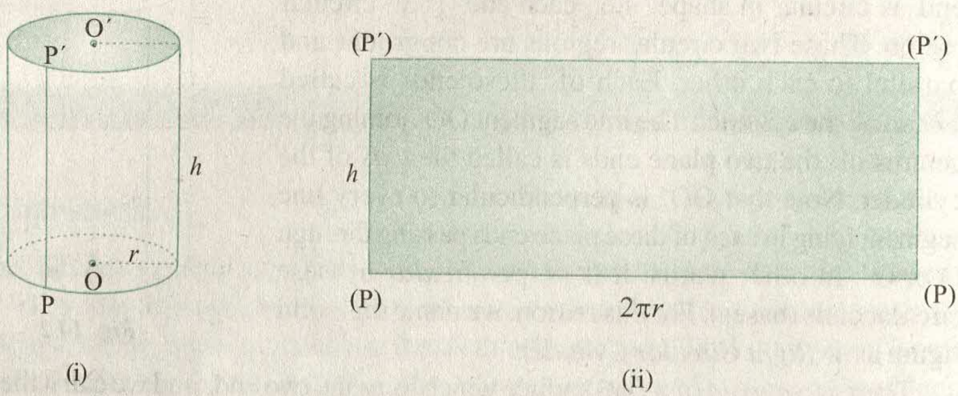


Fig. 14.3

Now let us consider the lateral (curved) surface. Does it have an area? If yes, how to determine it? If the curved surface could be somehow made flat, i.e., a plane region [Fig. 14.3 (ii)], then there would be no difficulty in determining the curved surface area. We proceed as follows :

We take a rectangular strip of paper of width h so that we can wrap it around the given cylinder of height h to just cover it.

Mark a generating line segment PP′ on the curved surface. Place the edge of the paper strip along PP′ and hold it fast. Now wrap the strip around the cylinder till you reach PP′ again. At this stage, cut off the strip along PP′. Now remove the cut off strip and spread it [Fig. 14.3 (ii)]. What do you observe? Of what shape is this cut off strip? It is a rectangle. What is the breadth of this rectangle? Clearly, it is h. What is the length of this rectangle? Note that the length of the rectangle has gone around the circular end of the cylinder just once. Also, the radius of the circular end is r. Thus, length of the rectangle is equal to the circumference of the circle with radius r. That is, the length of the rectangle is $2\pi r$.

Therefore, the area of the rectangle $= 2\pi r \times h = 2\pi rh$

As can be seen easily :

Area of the curved surface $=$ Area of the rectangle of length $2\pi r$ and breadth h

$= 2\pi rh$

Thus, for a cylinder of base radius r and height h:

Curved surface area = $2\pi rh$

Area of each end = πr^2

and total surface area = $2\pi rh + \pi r^2 + \pi r^2 = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$

Note that total surface area is for a solid cylinder.

Remark : These formulas can also be verified by taking a piece of paper and folding it in the form of a cylinder.

We now take some examples to illustrate the use of these formulas.

Example 1 : Find the curved surface area of a right circular cylinder of base radius 3 cm and height 5 cm. (Take $\pi = 3.14$.)

Solution : Curved surface area of a cylinder = $2\pi rh$

$$= 2 \times 3.14 \times 3 \times 5 \text{ cm}^2$$

$$= 94.2 \text{ cm}^2$$

Example 2 : The curved surface area of a right circular cylinder of height 14 cm is 88 cm^2. Find the radius of the base of the cylinder.

Solution : Curved surface area = $2\pi rh$

Therefore, $$88 = 2 \times \frac{22}{7} \times r \times 14$$

or $$r = \frac{88 \times 7}{2 \times 22 \times 14}$$

$$= 1$$

Thus, the radius of the base is 1 cm.

Example 3 : It is required to make a closed cylindrical tank of height 1 m and base diameter 140 cm from a metal sheet. How many square metres of the metal sheet are required for the same?

Solution : Here, diameter = 140 cm

$$\therefore \quad \text{Radius } r = \frac{140}{2} \text{ cm} = 70 \text{ cm} = \frac{70}{100} \text{ m} = \frac{7}{10} \text{ m.}$$

Height $h = 1$ m

Total surface area of the tank = $2\pi r(h+r)$

$$= 2\times\frac{22}{7}\times\frac{7}{10}\left(1+\frac{7}{10}\right)\text{m}^2$$

$$= \frac{2\times 22\times 17}{100}\text{m}^2 = 7.48 \text{ m}^2$$

Thus, the area of the metal sheet required is 7.48 m^2.

Hence, 7.48 square metres of metal sheet are required.

Example 4 : Inner and outer radii of a metallic pipe are 3 cm and 3.5 cm respectively, (Fig.14.4). If the length of the pipe is 56 cm, find its total surface area. (Use $\pi = 3.14$.)

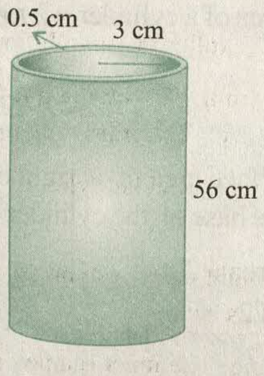


Fig. 14.4

Solution : Inner curved surface area of the pipe = $2\pi rh = 2\pi \times 3 \times 56$ cm^2 (1)

Outer curved surface area of the pipe = $2\pi \times 3.5 \times 56$ cm^2 (2)

Area of the two ends = $2\pi\,[(3.5)^2 - (3)^2]$ cm^2 = $2\pi \times 6.5 \times 0.5$ cm^2 (3)

$\therefore$ Total surface area = $[(2\pi \times 3 \times 56) + (2\pi \times 3.5 \times 56) + (2\pi \times 6.5 \times 0.5)]$ cm^2

[From (1), (2) and (3)]

$$= \pi\,(336 + 392 + 6.5) \text{ cm}^2$$

$$= \pi\,(734.5) \text{ cm}^2$$

$$= 3.14 \times 734.5 \text{ cm}^2$$

$$= 2306.33 \text{ cm}^2$$

EXERCISE 14.1

1. A right circular cylinder has base radius 8 cm and height 35 cm. Find the curved surface area of the cylinder.

2. The circumference of the base of a right circular cylinder is 176 cm. If the height of the cylinder is 1 m, find the lateral surface area of the cylinder.

3. A closed circular cylinder has diameter 10 cm and height 15 cm. Find the total surface area of the cylinder. (Use $\pi = 3.14$.)

4. The radius of the base of a closed right circular cylinder is 21 cm and its height is 1 m. Find the total surface area of the cylinder.

5. The diameter of a roller is 84 cm and its length is 120 cm. It takes 500 complete revolutions to move once over to level a playground. Find the area of the playground in m^2.
   [*Hint :* Area levelled by the roller in 1 revolution = curved surface area of the roller.]

6. A cylindrical pillar is 50 cm in diameter and 3.5 m in height. Find the cost of white-washing the curved surface of the pillar at the rate of Rs 12.50 per m^2.

7. Curved surface area of a right circular cylinder of height 35 cm is 121 cm^2. Find the radius of its base.

8. Curved surface area of a right circular cylinder is 4.4 m^2. If the radius of the base of the cylinder is 0.7 m, find its height.

9. A metal pipe is 77 cm long. The inner diameter of a cross-section is 4 cm, the outer diameter being 4.8 cm. Find its
   (i) inner curved surface area.
   (ii) outer curved surface area.
   (iii) total surface area.

10. The inner diameter of a circular well is 3.5 m. It is 10 m deep. Find its
    (i) inner curved surface area.
    (ii) the cost of plastering the inner curved surface at the rate of Rs 40 per m^2.

11. The outer diameter of a pipe is 1 m and it is 21 m long. Find :
    (i) its outer curved surface area.
    (ii) the cost of painting the outer surface of the pipe at the rate of Rs 25 per m^2.

12. A cylindrical vessel open at the top has a base diameter 21 cm and height 14 cm. Find the cost of tin-plating its inner part at the rate of Rs 5 per 100 cm^2.

14.4 Right Circular Cone

An icecream cone, a joker's cap, a conical vessel, a conical tent, etc. (Fig. 14.5) are some of the objects from daily life which suggest or bring to our mind the concept of a right circular cone, which is a geometric figure.

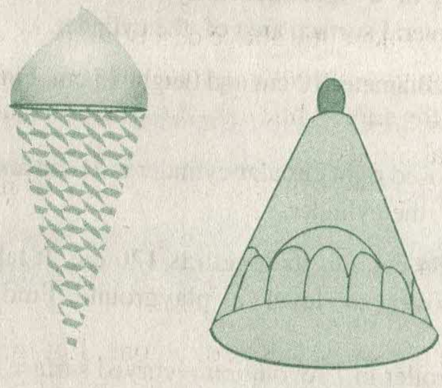


Fig. 14.5

In Fig. 14.6, outline of a right circular cone is given. It will help us to describe a right circular cone in geometric terms already known to us as follows :

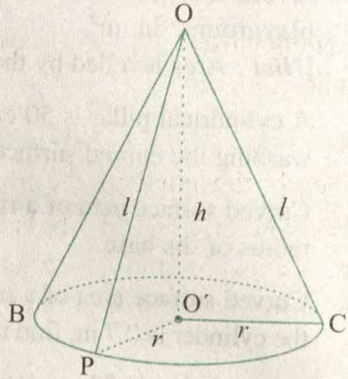


Fig. 14.6

A right circular cone has a *plane end* which is circular in shape, i.e., the plane end of a right circular cone is a *circular region*. This end is called the *base* of the cone. Its radius r is called the *radius* of the base of the cone or simply the base radius.

There is also a corner, which is a point (O in Fig. 14.6) of the cone farthest from the base. It is called the *vertex* of the cone. Connecting the vertex and the circular edge of the base, there is a *curved surface*. This is also referred to as the *lateral surface* of the cone.

If O is the vertex, and O′ is the centre of the base, then OO′ is called the *axis* of the cone and the length of the line segment OO′ is called the *height* of the cone. Observe that OO′ is *perpendicular* to the base. For this reason, we name this solid as a *right circular cone*.

As in the case of a cylinder, here also the radius r of the base and the height h of the cone determine the *size* of the cone.

You can also observe that for each point P on the circular edge of the base, the line segment OP lies on the curved surface of the cone. The length l of each of such line segments is called the *slant height* of the cone.

Note that $\Delta OO'P$ is a right triangle, right angled at O′ (Fig. 14.6).

Therefore, we have

$$l^2 = r^2 + h^2 \qquad \text{(By Pythagoras Theorem)}$$

or $$l = \sqrt{r^2 + h^2}$$

Remark : As in the case of a cylinder, the above description of a cone brings to our mind, two distinct but related figures – the *hollow cone* and *the solid cone*. The hollow cone is just the lateral surface of the cone, while the solid cone contains the interior of the cone also in addition to the lateral surface.

Further, unless stated otherwise, we shall often use the word 'cone' to mean a 'right circular cone'.

14.5 Surface Area of a Right Circular Cone

Let us now try to determine the surface area of a cone. For this, we perform the following activity :

Activity 2 : Take a right circular cone of base radius r and height h [Fig. 14.7 (i)]. Clearly, the length of the circular edge is $2\pi r$ and the area of the base = πr^2.

Now let us consider the lateral (curved) surface. Does it have an area? How to determine it? If somehow, the curved surface could be made flat, i.e., a plane region, then there will be no difficulty in determining the curved surface area. For this, we proceed as follows :

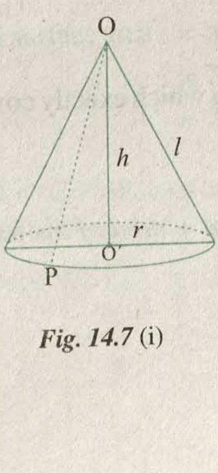


Fig. 14.7 (i)

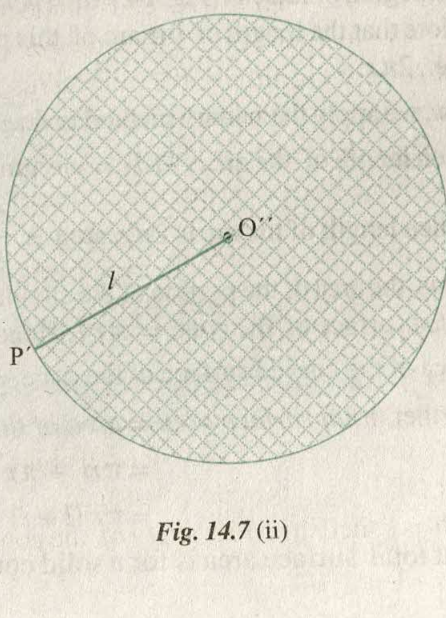
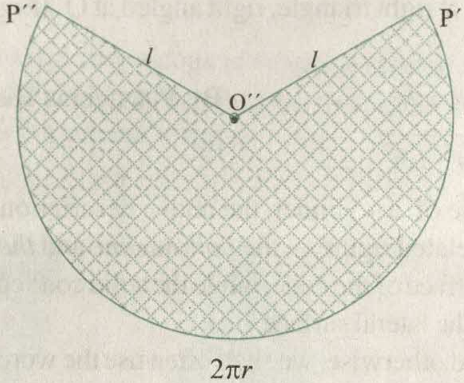


Fig. 14.7 (ii)



(iii)

Fig. 14.7

Let P be a point on the circular edge of the cone. Let OP = l. Now draw a circle of radius l with centre O″ on a sheet of paper [Fig. 14.7 (ii)]. Cut out the sheet along the circle. Thus, we obtain a circular sheet (disc) of paper of radius l and centre O″. Cut this disc along the radius P′O″.

Now we place the radius O″P′ of the disc along OP with O″ at O, P′ at P and wrap the disc around the cone, keeping O″ fixed at O and P′ fixed at P. When we come back to OP, we cut off the remaining part of the disc. Thus, we obtain a part or portion of a circular region of radius l [Fig. 14.7 (iii)] which exactly covers the curved surface of the cone. Note that the length of the arc of this portion is equal to the circular edge of the cone, i.e., $2\pi r$.

Now, we apply the unitary method to determine the curved surface area as follows :

When length of the arc is $2\pi l$ (= circumference of the disc), area = πl^2

∴ When length of the arc is $2\pi r$, area = $\dfrac{\pi l^2}{2\pi l} \times 2\pi r = \pi r l$

Thus, the area of the portion of the circular disc of radius l which exactly covers the curved surface of the cone = $\pi r l$. Hence,

Area of the curved surface of the cone = $\pi r l$

Further, *total surface area = area of the curved surface + area of the base*

$$= \pi r l + \pi r^2$$

$$= \pi r (l + r)$$

Note that total surface area is for a solid cone.

Remark : The above formulas can also be verified by taking a part of a circular region made of paper [Fig. 14.7 (iii)] and folding it in the form of cone.

Now we take some examples to illustrate the use of these formulae.

Example 5 : Diameter of the base of a cone is 10.5 cm and its slant height is 10 cm. Find the curved surface area of the cone.

Solution : Diameter of the base = 10.5 cm

Therefore, base radius $r = \frac{10.5}{2}$ cm

slant height $l = 10$ cm (Given)

Hence, curved surface area = πrl

$$= \frac{22}{7} \times \frac{10.5}{2} \times 10 \text{ cm}^2 = \frac{22}{7} \times \frac{105}{20} \times 10 \text{ cm}^2$$

$$= 165 \text{ cm}^2$$

Example 6: The height of a cone is 16 cm and the radius of its base is 12 cm. Find the curved surface area and the total surface area of the cone. (Use $\pi = 3.14$.)

Solution : Here, $h = 16$ cm and $r = 12$ cm. Let the slant height of the cone be l.

Then, from $l^2 = r^2 + h^2$, we have

$$l = \sqrt{r^2 + h^2}$$

$$= \sqrt{12^2 + 16^2} \text{ cm} = 20 \text{ cm}$$

Hence, curved surface area = $\pi rl = 3.14 \times 12 \times 20 \text{ cm}^2$

$$= 753.6 \text{ cm}^2$$

Total surface area $= \pi rl + \pi r^2$

$$= 753.6 \text{ cm}^2 + 3.14 \times 144 \text{ cm}^2$$

$$= 753.6 \text{ cm}^2 + 452.16 \text{ cm}^2 = 1205.76 \text{ cm}^2$$

Example 7 : Curved surface area of a cone is 308 cm^2 and its slant height is 14 cm. Find (i) radius of the base, and (ii) total surface area of the cone.

Solution : (i) We have $\pi r l = 308$

or $\frac{22}{7} \times r \times 14 = 308$

or $$r = \frac{308 \times 7}{22 \times 14} = 7$$

Thus, radius of the base is 7 cm.

(ii) Total surface area $= \pi rl + \pi r^2 = (308 + \frac{22}{7} \times 49)\ \text{cm}^2 = (308 + 154)\ \text{cm}^2$

$$= 462\ \text{cm}^2$$

Example 8 : The radius of the base of a conical tent is 12 m. The tent is 9 m high. Find the cost of the canvas required to make the tent, if one square metre of canvas costs Rs 120. (Take $\pi = 3.14$.)

Solution : Let the slant height of the cone be l. Then, from $l^2 = r^2 + h^2$, we have

$$l = \sqrt{144 + 81}\ \text{m} = 15\ \text{m}$$

Therefore, curved surface area of the tent $= \pi rl = 3.14 \times 12 \times 15\ \text{m}^2$

$$= 565.2\ \text{m}^2$$

Hence, the canvas required to make the tent is $565.2\ \text{m}^2$.

Cost of the canvas at Rs 120 per m^2 = Rs 120×565.2 = Rs 67824

EXERCISE 14.2

1. Find the curved surface area of a right circular cone whose slant height is 10 cm and base radius 7 cm.

2. The diameter of a right circular cone is 14 cm and its slant height is 9 cm. Find its
   (i) curved surface area. (ii) total surface area

3. Find the curved surface area of a cone, if its slant height is 60 cm and the radius of its base is 25 cm. (Take $\pi = 3.14$.)

4. Find the total surface area of a cone, if its slant height is 24 dm and diameter of its base is 14 dm.

5. The radius of the base of a cone is 9 cm and its height is 12 cm. Find the curved surface area of the cone.
   [*Hint :* First find the slant height.]

6. A conical tent is 10 m high and the radius of its base is 24 m. Find
   (i) slant height of the tent.
   (ii) cost of the canvas required to make the tent, if the cost of 1m^2 canvas is Rs 90.

7. The curved surface area of a right circular cone is 528 cm^2. If the slant height of the cone is 21 cm, find the
   (i) radius of the base. (ii) total surface area of the cone.
8. What length of tarpaulin 3 m wide will be required to make a conical tent of height 8 m and base radius 6 m? (Use $\pi = 3.14$.)
9. The slant height and the base diameter of a conical tomb are 25 m and 14 m, respectively. Find the cost of white washing its curved surface at the rate of Rs 210 per 100 m^2.
10. An open metallic conical tank is 4 m deep and its circular top has diameter 6 m. Find the area of the metallic sheet required to make the tank. (Use $\pi = 3.14$.)
11. A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.
12. Curved surface area of an icecream cone of slant height 12 cm is 113.04 cm^2. Find the base radius of the cone. (Use $\pi = 3.14$.)

14.6 Sphere

A ball, a cricket ball, a marble, etc. (Fig. 14.8) are some of the objects from daily life which suggest or bring to our mind the concept of a sphere, which is a geometric figure.

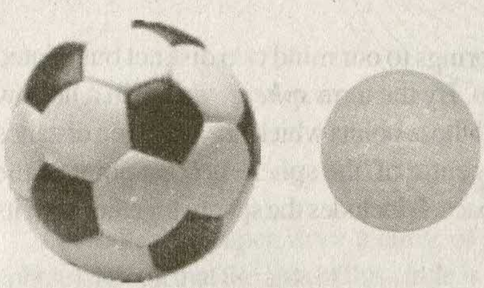


Fig. 14.8

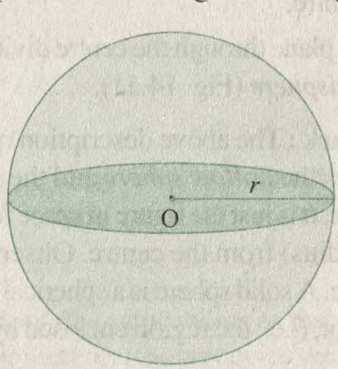


Fig. 14.9

In Fig. 14.9, an outline of a sphere is given. It will help us to describe the sphere in geometric terms, already known to us. A sphere is a figure formed by all those points in space which are at the same distance from a fixed point. The fixed point (in this figure point O) is called the *centre* of the sphere and the 'same distance' is called the *radius* of the sphere. A line segment passing through the centre of the sphere and having its end points on the sphere is called a *diameter* of the sphere. As in the case of a circle, the length of a

diameter is also called the diameter of the sphere. It is clear that the diameter d and the radius r of the sphere are related by the relation $d = 2r$.

The radius r of a sphere completely determines the size of the sphere.

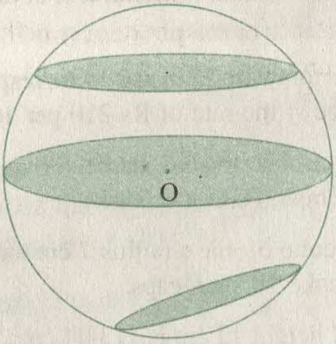


Fig. 14.10

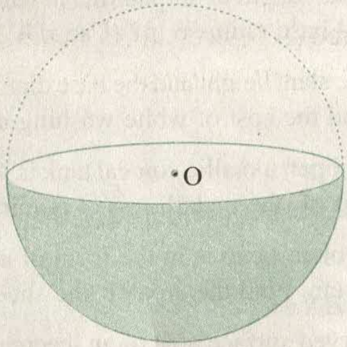


Fig. 14.11

A section of a sphere by a plane is a circle (Fig. 14.10). The plane through the centre gives the largest circular section of the sphere. The radius of this largest section is the same as that of the sphere. The circular section gets smaller and smaller as we move away from the centre.

A plane through the centre divides the sphere into two equal parts. Each part is called a *hemisphere* (Fig. 14.11).

Remark : The above description of a sphere brings to our mind two distinct but related figures the *hollow sphere* and the *solid sphere*. By the term *sphere*, we mean a hollow sphere. It is just the figure in space formed by all those points which are at a given distance (= radius) from the centre. Observe that the centre of the sphere is not a point of the sphere. A solid sphere is a spherical region in space. It includes the sphere together with its interior, (i.e., the region enclosed by it).

14.7 Surface Area of a Sphere

Let us now try to determine the surface area of a sphere. For this, we perform some activities.

Activity 3 : Consider a sphere of centre O and radius r. Let a plane through O divide the sphere into two hemispheres [Fig. 14.12 (i)].

Let us consider the curved surface of the upper hemisphere. Does it have an area? If the curved surface, somehow, could be made flat, i.e., a plane region, then there would be no difficulty. We could do it easily for a cylinder and for a cone by wrapping a sheet of

paper around each. However, this is not possible here. We, therefore, adopt a different procedure here as given below :

Take a long string. Starting from the uppermost point of the hemisphere, wrap it along the hemisphere, in the form of a spiral [Fig. 14.12 (ii)]. Continue till the entire hemisphere is covered (A light coating of gum on the surface of the hemisphere may be helpful in keeping the string in place.) Now unwrap and measure the length of the string.

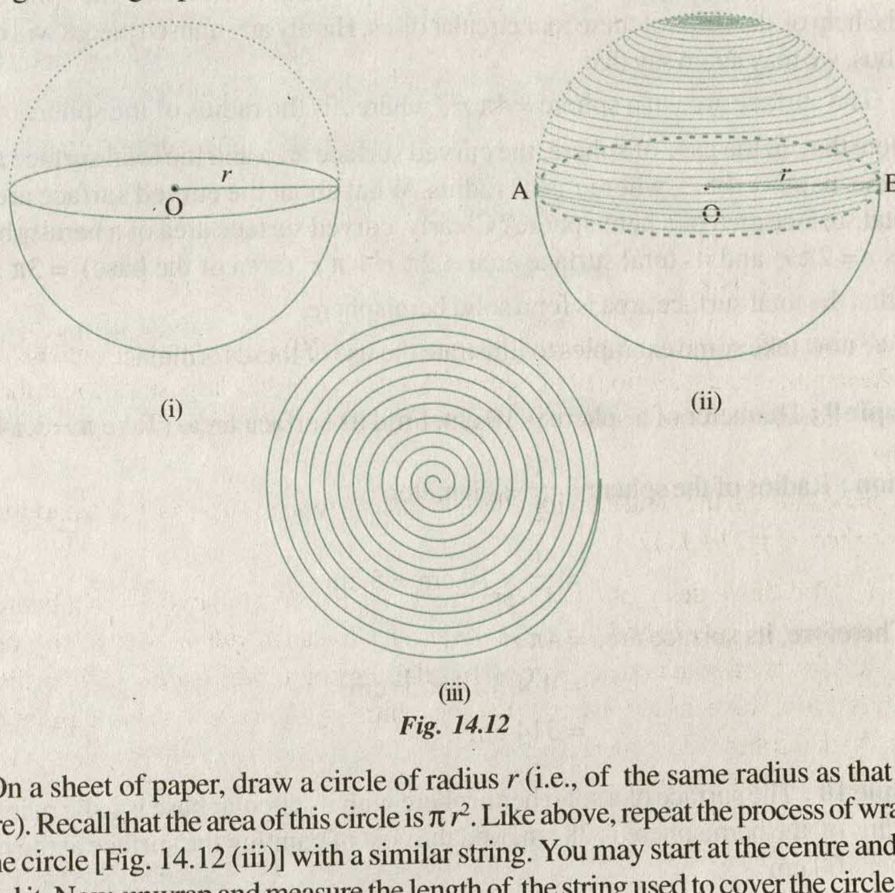


Fig. 14.12

On a sheet of paper, draw a circle of radius r (i.e., of the same radius as that of the sphere). Recall that the area of this circle is πr^2. Like above, repeat the process of wrapping for the circle [Fig. 14.12 (iii)] with a similar string. You may start at the centre and spiral around it. Now, unwrap and measure the length of the string used to cover the circle. What do you observe? You will observe that the length of the string needed to cover the hemisphere is nearly *twice* the length of the string needed to cover the circular region. The slight difference is due to small gaps in wrapping and covering. As thickness of the string is the same in both cases, therefore, surface area of the hemisphere = 2 × (area of the circle) = $2\pi r^2$.

Hence, the surface area of the sphere = 2 × (area of one hemisphere)

$$= 2 \times 2\pi r^2 = 4\pi r^2$$

Activity 4 : Take a sphere of radius r. From a sheet of paper, cut out *four circular discs each of radius r*. Clearly, the area of each disc is πr^2.

Now cut the four circular discs into thin strips and try to cover the surface of the sphere with the same. Try placing the strips as closely as possible, leaving no gaps as far as possible. You may cut some strips into smaller pieces if needed. What do you observe? You will observe that you would be able to cover the entire surface of the sphere once with the help of the strips of these four circular discs. Hardly any unused pieces will be left.

Thus, we may again say that

The surface area of a sphere $= 4\pi r^2$, where r is the radius of the sphere.

Note that, in the case of sphere, the curved surface area and the total surface area is the same, namely $4\pi r^2$, where r is its radius. What about the curved surface area and the total surface area of a hemisphere? Clearly, curved surface area of a hemisphere of radius $r = 2\pi r^2$ and its total surface area $= 2\pi r^2 + \pi r^2$ (area of the base) $= 3\pi r^2$.

Note that the total surface area is for a solid hemisphere.

We now take some examples to illustrate the use of these formulas.

Example 9 : Diameter of a sphere is 10 cm. Find its surface area. (Take $\pi = 3.14$.)

Solution : Radius of the sphere $= \frac{1}{2} \times$ diameter

$$= \frac{1}{2} \times 10 \text{ cm} = 5 \text{ cm}$$

Therefore, its surface area $= 4\pi r^2$

$$= 4 \times 3.14 \times 5^2 \text{ cm}^2$$

$$= 314 \text{ cm}^2$$

Example 10 : The surface of a solid hemisphere with its circular base is to be painted. If the radius of the hemisphere is 28 cm, find the cost of painting the surface at the rate of Rs 3 per 100 cm^2.

Solution : Surface area of the hemisphere $= 3\pi r^2$

$$= 3 \times \frac{22}{7} \times 28 \times 28 \text{ cm}^2$$

$$= 7392 \text{ cm}^2$$

Therefore, cost of painting the surface at the rate of Rs 3 per 100 cm^2

$$= \text{Rs } \frac{3}{100} \times 7392$$
$$= \text{Rs } 221.76$$

Example 11 : Assuming the Earth to be a sphere of radius 6370 km, find

(i) surface area of the Earth.

(ii) area of the land, if three-fourths of the Earth's surface is covered by water.

Solution : (i) Surface area of the Earth = $4\pi r^2$

$$= 4 \times \frac{22}{7} \times 6370 \times 6370 \text{ km}^2$$
$$= 510109600 \text{ km}^2$$

(ii) $\frac{3}{4}$ of the Earth's surface is covered with water.

Therefore, area of the land = $\frac{1}{4} \times$ (surface area of the Earth)

$$= \frac{1}{4} \times 510109600 \text{ km}^2$$
$$= 127527400 \text{ km}^2$$

EXERCISE 14.3

1. Find the surface area of a sphere whose diameter is
   (i) 14 cm (ii) 21 cm (iii) 3.5 m
2. Find the surface area of a sphere whose radius is
   (i) 10.5 cm (ii) 5.6 m (iii) 14 cm
3. The radius of a spherical balloon increases from 7 cm to 14 cm as air is being pumped into it. Find the ratio of surface areas of the balloon in the two cases.
4. A hemispherical bowl made of brass has inner diameter 10.5 cm. Find the cost of tin-plating it on the inside at the rate of Rs 16 per 100 cm^2.
5. The dome of a building is in the form of a hemisphere. Its radius is 6.3 m. Find the cost of painting it at the rate of Rs 12 per m^2.
6. Find the radius of a sphere whose surface area is 154 cm^2.

7. The diameter of the Moon is approximately one fourth the diameter of the Earth. Find the ratio of their surface areas.

8. Radius of a shotput is 7 cm. Find its surface area.

9. A hemispherical bowl is made of steel, 0.25 cm thick. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.

10. A right circular cylinder just encloses a sphere of radius r (Fig. 14.13). Find
    (i) surface area of the sphere.
    (ii) curved surface area of the cylinder.
    (iii) ratio of the areas obtained in (i) and (ii).

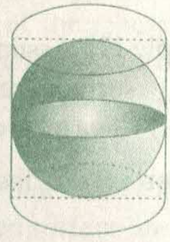


Fig. 14.13

Things to Remember

[*Throughout r, h and l have the usual meaning.*]

1. Each plane end of a right circular cylinder is called its base.
2. The line segment joining the centres of the two circular ends of a cylinder is called the axis of the cylinder. The length of the axis is called the height of the cylinder.
3. The curved surface which joins the two bases of a right circular cylinder is called its lateral surface.
4. Lateral or curved surface area of a right circular cylinder = $2\pi rh$.
5. Total surface area of a right circular cylinder = $2\pi rh + 2\pi r^2 = 2\pi r(h + r)$.
6. The plane end of a right circular cone is called its base. The only corner of the cone is called the vertex of the cone.
7. The line segment joining the vertex of the cone to the centre of its base is called the axis of the cone. The length of the axis is called the height of the cone.
8. The length of the line segment joining the vertex of the cone to a point on the circular edge is called its slant height. Further, $l = \sqrt{r^2 + h^2}$.
9. The curved surface connecting the vertex to the circular edge of the right circular cone is called its lateral surface.

10. Lateral or curved surface area of a cone = πrl.

11. Total surface area of a cone = $\pi rl + \pi r^2 = \pi r(l + r)$.

12. A sphere with centre O and radius r is the figure (in space) consisting of all the points (in space) which are at a distance r from O.

13. Surface area of a sphere of radius r is $4\pi r^2$.

14. Lateral or curved surface area of a hemisphere of radius r is $2\pi r^2$.

15. Total surface area of a hemisphere of radius r is $2\pi r^2 + \pi r^2$, i.e., $3\pi r^2$.

CHAPTER

15

VOLUMES

15.1 Introduction

We know that a solid occupies some region in space and the magnitude of this region is called the *volume* of the solid. One of the standard units of volume is cubic centimetre (cu cm) or a centimetre cube (cm^3). The unit 1 cubic centimetre is the volume of a cube of side 1 cm. The volume of a cube of side a cm is a cm $\times$ a cm $\times$ a cm, i.e., a^3 cm^3 (or a^3 cu cm).

Some other standard units of volume are mm^3, dm^3, m^3 and km^3. The capacity of a container with volume 1000 cm^3 is 1 litre or 1 *l*.

We also know that the volume V of a cuboid of sides l cm, b cm and h cm is given by

$$V = l \times b \times h \text{ cm}^3$$

In this Chapter, we shall study volumes of some other familiar solids, viz., the cylinder, the cone and the sphere. As in Chapter 14, we shall take the value of π as $\frac{22}{7}$, unless stated otherwise.

15.2 Volume of a Right Circular Cylinder

We have studied this solid in the previous chapter where we considered its surface area. We now consider its volume. Let r be the base radius and h the height of the cylinder.

Let us first consider a cuboid with length, breadth and height equal to a, b and c units, respectively. We know that the volume of the cuboid is $a \times b \times c$ unit^3. We also know that the lateral surface area of the cuboid (in square units) is $2(a \times c + b \times c)$. The surface area can be expressed as $2(a + b) \times c$. Also, $2(a + b)$ is the perimeter of the base. How are the two related?

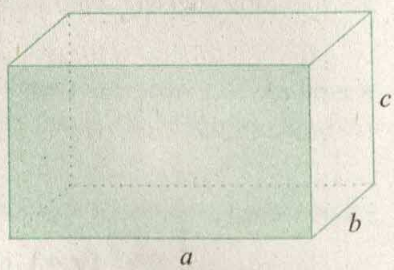


Fig. 15.1

Lateral surface area of a cuboid = Perimeter of the base $\times$ Height.

Now consider its volume. We know that the volume is $a \times b \times c$ and $a \times b$ is the area of its base. Hence,

Volume of the cuboid = $(a \times b) \times c$ = Area of the base × Height.

Thus, for a cuboid :

Lateral Surface Area = Perimeter of the Base × Height

Volume = Area of the base × Height

Also, for a cylinder,

Lateral (curved) surface area = $2\pi rh$ = Perimeter of the Base × Height

It is, therefore, natural to expect that

Volume of the cylinder = Area of the base × Height = $\pi r^2 h$

This result is true, but formal proof of this result is out of the scope of this book. However, we may verify this result by an experiment.

Activity 1 : Take a cylindrical vessel. Measure its radius r and height h in cm and enter these in the table below. Compute the value $\pi r^2 h$ by taking $\pi = \frac{22}{7}$ or 3.14.

Cylinder	r	h	$\pi r^2 h$	V	$V - \pi r^2 h$
I					
II					
III					

Put this value in the corresponding column of the table. Now fill up the vessel with water. The volume of the water in this case is equal to the volume V of the cylinder (cylindrical vessel). Measure the volume of this water using a measuring flask. This volume will be in the unit ml. Using the relation 1 ml = 1 cm^3, convert this measure in cm^3 and write it in the next column. In the final column, write the difference $V - \pi r^2 h$. Repeat this experiment with at least two more cylindrical vessels of different radii and heights.

What do we observe? We observe that the entries in the column headed by $V - \pi r^2 h$ are nearly zero. Thus, we can say that $V - \pi r^2 h = 0$, i.e.,

$$V = \pi r^2 h = (\text{Area of the base}) \times \text{Height}$$

Let us illustrate the use of this formula by taking some examples.

Example 1 : The diameter of the base of a right circular cylinder is 7 cm. If its height is 40 cm, find its volume.

Solution : Since the diameter of the base is 7 cm, its radius $r = \frac{7}{2}$ cm. Also, $h = 40$ cm, and $\pi = \frac{22}{7}$. Therefore, the volume of the cylinder is given by

$$V = \pi r^2 h = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 40 \text{ cm}^3 = 1540 \text{ cm}^3$$

Example 2 : A metallic cylindrical pipe has thickness 0.5 cm and outside diameter 4.5 cm. If 1 cm^3 of the metal has mass 8 g, find the mass of a 77 cm long pipe.

Solution : We first find the volume of the metallic part of the cylindrical pipe. This is the difference of the volume of two solid cylinders one of diameter 4.5 cm and the other of diameter (4.5 – 1.0) cm or 3.5 cm.

For outer cylinder, radius $= \frac{1}{2} \times 4.5$, and height = 77 cm

$$\therefore \quad \text{Volume} = \frac{22}{7} \times \frac{4.5}{2} \times \frac{4.5}{2} \times 77 \text{ cm}^3 = 242 \times \left(\frac{4.5}{2}\right)^2 \text{ cm}^3 \qquad (1)$$

For inner cylinder, radius $= \left(\frac{4.5}{2} - 0.5\right)$ cm $= \frac{3.5}{2}$ cm, and height = 77 cm

$$\therefore \quad \text{Volume} = \frac{22}{7} \times \frac{3.5}{2} \times \frac{3.5}{2} \times 77 \text{ cm}^3 = 242 \times \left(\frac{3.5}{2}\right)^2 \text{ cm}^3 \qquad (2)$$

$\therefore$ The required volume of the pipe = Volume of the outer cylinder – Volume of the inner cylinder

$$= 242 \times \left[\left(\frac{4.5}{2}\right)^2 - \left(\frac{3.5}{2}\right)^2\right] \text{ cm}^3 \qquad \text{[From (1) and (2)]}$$

$$= 242 \times \left(\frac{4.5}{2} + \frac{3.5}{2}\right) \times \left(\frac{4.5}{2} - \frac{3.5}{2}\right) \text{ cm}^3$$

$$= 242 \times 4 \times 0.5 \text{ cm}^3 = 484 \text{ cm}^3$$

$\therefore$ Mass of the pipe $= 484 \times 8$ g [Since 1 cm^3 of the metal has mass 8 g]

$= 3872$ g

$= 3.872$ kg

Example 3 : A $11 \text{ cm} \times 4 \text{ cm}$ rectangular piece of paper is folded and taped without overlapping to make a cylinder of height 4 cm (Fig. 15.2). Find the volume of the cylinder so formed.

Solution : Note that the longer edges of the piece of paper become the circular edges of the cylinder. Hence, the circumference of each circular edge of the cylinder is 11 cm. Let r be the radius of each circular edge and h, the height of the cylinder.

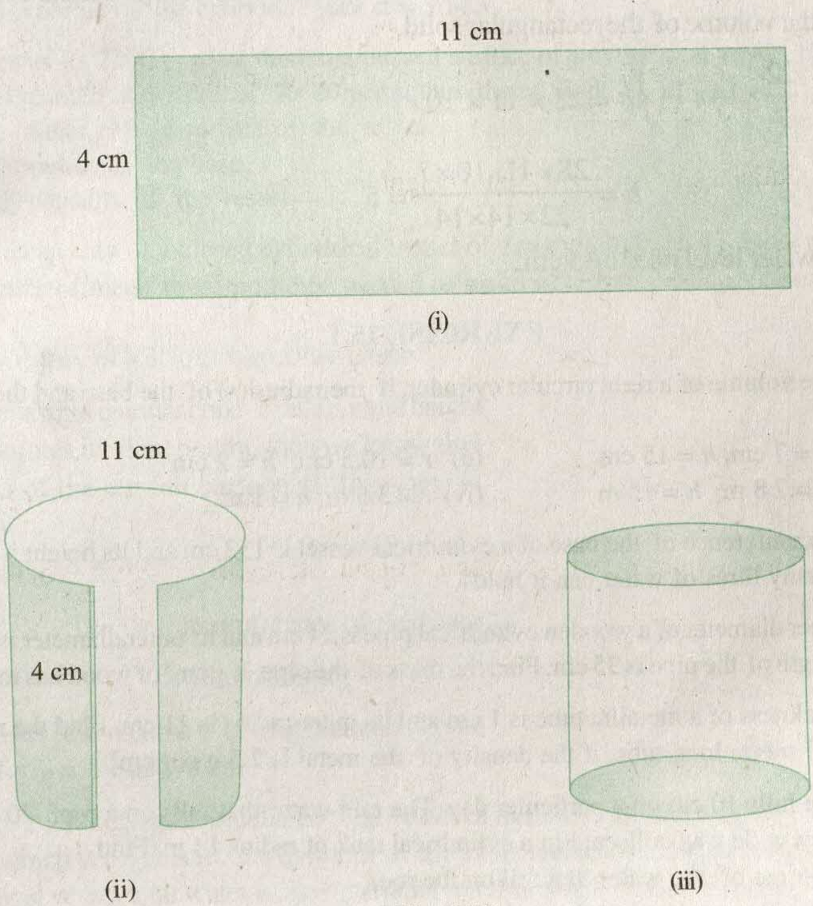


Fig. 15.2

Then $2\pi r = 11$ cm

$\therefore$ $$r = \frac{11}{2\pi}\text{ cm} = \frac{11\times 7}{2\times 22}\text{ cm} = \frac{7}{4}\text{ cm}$$

$\therefore$ Volume of the cylinder $= \pi r^2 h = \frac{22}{7}\times\frac{7}{4}\times\frac{7}{4}\times 4\text{ cm}^3 = 38.5\text{ cm}^3$

Example 4 : A cylindrical bucket, 14 cm in radius, is filled with water to some height. If a rectangular solid of size 28 cm × 11 cm × 10 cm is immersed in the water, find the height by which water rises in the bucket.

Solution : If h cm is the rise in water level, then the volume of the cylindrical column of height h = the volume of the rectangular solid,

i.e., $$\frac{22}{7}\times 14\times 14\times h = 28\times 11\times 10$$

or $$h = \frac{28\times 11\times 10\times 7}{22\times 14\times 14} = 5$$

Hence, the water level rises by 5 cm.

EXERCISE 15.1

1. Find the volume of a right circular cylinder, if the radius (r) of the base and the height (h) are :
   (i) $r = 7$ cm, $h = 15$ cm (ii) $r = 10.5$ cm, $h = 2$ cm
   (iii) $r = 2.8$ m, $h = 15$ m (iv) $r = 3.5$ m, $h = 1$ m

2. The circumference of the base of a cylindrical vessel is 132 cm and its height is 25 cm. How many litres of water can it hold?

3. The inner diameter of a wooden cylindrical pipe is 24 cm and its outer diameter is 28 cm. The length of the pipe is 35 cm. Find the mass of the pipe, if 1 cm^3 of wood has mass 3 g.

4. The thickness of a metallic tube is 1 cm and its outer radius is 11 cm. Find the mass of such a 1 metre long tube, if the density of the metal is 7.5 g per cm^3.

5. The rain falls 10 cm on a particular day. The rain water that falls on a roof 70 m long and 44 m wide was collected in a cylindrical tank of radius 14 m. Find
   (i) volume of the water that fell on the roof,
   (ii) rise of water level in the tank due to rain water.

6. A circular well of radius 3.5 m is dug 20 m deep, and the earth so dug is spread out on a rectangular plot of length 14 m and breadth 11m. Find
(i) volume of the earth dug out,
(ii) area of the rectangular plot,
(iii) height of the platform formed by spreading the earth on the rectangular plot.

7. A soft drink is available in two packs: a tin can with a rectangular base of length 5 cm and width 4 cm and having a height of 15 cm, and a plastic cylinder with circular base of diameter 7 cm and height 10 cm. Which container has greater capacity and by how much?

8. If the lateral surface of a cylinder of height 5 cm is 94.2 cm^2, then find
(i) radius of the base,
(ii) volume of the cylinder. (Take $\pi = 3.14$.)

9. It costs Rs 2200 to paint the inner curved surface of a cylindrical vessel 10 m deep. If it is painted at the rate of Rs 20 per square metre, find
(i) inner curved surface of the vessel,
(ii) radius of the base,
(iii) capacity of the vessel.

10. The capacity of a closed cylindrical vessel of height 1 m is 15.4 l. How many square metres of metal sheet would be needed to make it?

15.3 Volume of a Right Circular Cone

Consider a right circular cone of radius r and height h. If l denotes its slant height, then we know that the area of the curved surface of this cone is given by

$$S = \pi r l$$

or

$$S = \frac{1}{2} \times \text{(circumference of the base)} \times \text{(slant height)}$$

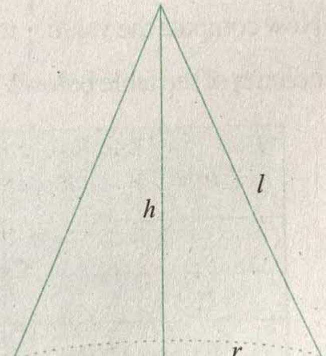


Fig. 15.3

To find the volume V of a right circular cone, we perform some activities.

Activity 2 : Take a right circular conical vessel of radius, say r, and height h. Take another vessel which is a right circular cylinder of the same radius r and same height h. Now fill the conical vessel with water to the brim and pour this water into the cylindrical vessel. The cylindrical vessel will not be full. In fact you can observe that the height of the water

column is even less than half the height of the vessel. So you may pour more water in it. Fill the conical vessel with water to the brim again and transfer the water to the cylindrical vessel. You may observe that the vessel is still not full. Once again fill the conical vessel and pour the water in the cylinder. Now the cylindrical vessel will be just full.

This experiment suggests that *the volume of the cylindrical vessel is three times the volume of the conical vessel* or *the volume of the cone* = $\frac{1}{3}$ × (*volume of the cylinder*). In other words, the volume V of a right circular cone of base radius r and height h is given by

$$V = \frac{1}{3}(\text{Volume of cylinder of base radius } r \text{ and height } h)$$

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}(\text{Area of Base}) \times \text{Height}$$

Here also, we shall not attempt to give a proof of this relation. However, like cylinder, we may verify the relation by taking cones of different radii and heights.

Activity 3 : Take a conical vessel and measure its radius r and height h in centimetres. Now compute the value $\frac{1}{3}\pi r^2 h$, by taking $\pi = \frac{22}{7}$ or 3.14. Put this value in the appropriate column of the table below :

Cone	r	h	$\frac{1}{3}\pi r^2 h$	V	$V - \frac{1}{3}\pi r^2 h$
I					
II					
III					

Now fill up this vessel with water to the brim and measure the volume V of this water in cm^3 using a measuring flask. Put this value in the next column. Now compute $V - \frac{1}{3}\pi r^2 h$ and enter it in the last column of the table. Repeat this experiment with at least two more conical vessels of different radii and heights.

What do we observe in the last column? The entries are nearly zero. Thus, we conclude that $V - \frac{1}{3}\pi r^2 h = 0$ Infact, we have the following formula :

$$\text{Volume V of a cone } = \frac{1}{3}\pi r^2 h$$

Example 5 : Find the volume of a right circular cone of height 2.04 m and base radius 14 cm.

Solution : Here, $r = 14$ cm and $h = 2.04$ m $= 204$ cm.

Volume of the cone is given by $V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \frac{22}{7} \times 14 \times 14 \times 204 \text{ cm}^3$

$$= 41888 \text{ cm}^3$$

Remark : A conical vessel of the above dimensions will hold 41.888 l of liquid.

Example 6 : The height and the slant height of a cone are 21 cm and 28 cm, respectively. Find: (i) the area of the base of the cone (ii) the volume of the cone.

Solution : (i) If r is the radius of the base of the cone, then we know that $r^2 = l^2 - h^2$, where h is the height and l is the slant height of the cone.

$\therefore$ $r^2 = 28^2 - 21^2$

$= (28 + 21)(28 - 21)$

$= (49)(7)$

or $r = 7\sqrt{7}$

Therefore, the area of the base A of the cone $= \pi r^2 = \frac{22}{7} \times \left(7\sqrt{7}\right)^2 \text{ cm}^2$

$$= \frac{22}{7} \times 49 \times 7 \text{ cm}^2 = 1078 \text{ cm}^2$$

(ii) The volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3} \times$ (area of the base) $\times$ height

$$= \frac{1}{3} \times 1078 \times 21 \text{ cm}^3 = 7546 \text{ cm}^3$$

Example 7 : The volume of a conical tent is 1232 m^3 and the area of its base is 154 m^2. Find the length of the canvas required to build the tent, if the canvas is 2 m in width.

Solution : The area of the canvas required to build the tent is equal to the curved surface area of the tent, i.e., $\pi r l$. We are given that

$$\text{Volume of the conical tent} = \frac{1}{3}\pi r^2 h = 1232 \text{ m}^3, \text{ and}$$

$$\text{Area of the base} = \pi r^2 = 154 \text{ m}^2 \qquad (1)$$

$$\therefore \qquad \frac{1}{3} \times 154 \times h = 1232$$

$$\text{or} \qquad h = \frac{1232 \times 3}{154} \text{ m} = 24 \text{ m} \qquad (2)$$

$$\text{Also, from (1)} \quad r = \sqrt{\frac{154 \times 7}{22}} \text{ m} = 7 \text{ m} \qquad (3)$$

$$\text{Therefore,} \qquad l = \sqrt{h^2 + r^2} = \sqrt{24^2 + 7^2} \text{ m} = 25 \text{ m} \qquad \text{[From (2) and (3)]}$$

$$\text{and} \qquad \pi r l = \frac{22}{7} \times 7 \times 25 \text{ m}^2 = 550 \text{ m}^2$$

$$\therefore \quad \text{Length of the canvas required} = \frac{\text{Surface area}}{\text{Width of canvas}} = \frac{550}{2} \text{ m} = 275 \text{ m}$$

EXERCISE 15.2

1. Find the volume of the right circular cone with
   (i) $r = 6$ cm and $h = 7$ cm (ii) $r = 3.5$ cm and $h = 12$ cm
2. Find the capacity of a conical vessel with
   (i) $r = 7$ cm and $l = 25$ cm (ii) $h = 12$ cm and $l = 13$ cm.
3. The height of a cone is 15 cm. If its volume is 1570 cm^3, find the area of the base. (Take $\pi = 3.14$.)
4. A metallic solid cone is melted and cast into the form of a circular cylinder of the same base as that of the cone. If the height of the cylinder is 7 cm, what was the height of the cone?
5. If the volume of a right circular cone of height 9 cm is 48π cm^3, find the diameter of its base.

6. A conical pit of top diameter 3.5 m is 12 m deep. What is its capacity in kilolitres?

7. The volume of a right circular cone is 9856 cm^3. If the diameter of the base is 28 cm, find
   (i) height of the cone,
   (ii) slant height of the cone,
   (iii) curved surface area of the cone.

8. A conical tent is 9 m high with base diameter 24 m. Find the number of persons it can accommodate if each person requires
   (i) 2 m^2 space on the ground,
   (ii) 15 m^3 space to breathe,
   (iii) 2 m^2 space on the ground and 15 m^3 space to breathe.

9. A right triangle ABC with its sides 5 cm, 12 cm and 13 cm is revolved about the side 12 cm. Find the volume of the solid so obtained (Fig. 15.4).

10. If the triangle ABC in Question 9 above is revolved about the side 5 cm, then find the volume of the solid so obtained. Find also the ratio of the volumes of the two solids obtained.

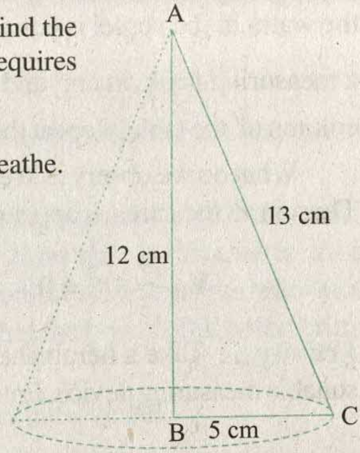


Fig. 15.4

15.4 Volume of a Sphere

We are all familiar with the solid called sphere. We also know that the surface area of a sphere of radius r is $4\pi r^2$. But we did not give any formal proof of this. In a similar manner, without giving any proof, we say that the volume V of a sphere of radius r is given by the formula:

$$\text{Volume V of a sphere} = \frac{4}{3}\pi r^3$$

However, we can experimentally verify this result.

Activity 4 : Take a solid ball (sphere) and measure its radius r using a suitable measuring device. Compute the value $\frac{4}{3}\pi r^3$ and enter it in the table below :

Sphere	r	$\frac{4}{3}\pi r^3$	V	$V - \frac{4}{3}\pi r^3$
I				
II				
III				

Now take a bucket large enough to hold the ball. Take another vessel large enough to keep the bucket so that the water overflown from the bucket may be collected. Keep the bucket in the vessel and fill it up to the brim with water. Now take the ball and put it gently into the bucket. Since the bucket is full to the brim, the water displaced by the ball will overflow in the large vessel. Remove the bucket (with ball) from the vessel. The volume of the water in the vessel is equal to the volume of the ball. Measure this volume V, using a measuring flask, in cm^3 and enter it in the table. Enter the value $V - \frac{4}{3}\pi r^3$ in the last column of the table. Repeat the experiment with at least two more balls of different radii.

What do we observe? We observe that the entries in the last column are nearly zero. Thus, in all the cases, it appears that

$$V - \frac{4}{3}\pi r^3 = 0 \quad \text{or} \quad V = \frac{4}{3}\pi r^3$$

Activity 5 : Take a hemispherical bowl and determine its radius r in cm using some suitable measuring device. Enter this value in the table below :

Hemispherical bowl	r	$\frac{2}{3}\pi r^3$	V	$V - \frac{2}{3}\pi r^3$
I				
II				
III				

Compute the value $\frac{2}{3}\pi r^3$ and enter it in the next column. Now fill the bowl with water upto the brim and then pour this water in a measuring flask to determine its volume V in cm^3. Enter this value in the column headed by V. In the next column enter the value $V - \frac{2}{3}\pi r^3$. Repeat the above experiment with at least two more hemispherical bowls with different radii.

What do we observe? We observe that the values in the column $V - \frac{2}{3}\pi r^3$ are nearly zero. Thus, we may say that

$$V - \frac{2}{3}\pi r^3 = 0 \text{ or } V = \frac{2}{3}\pi r^3$$

In other words, the volume of the water in the hemispherical bowl of radius r is $\frac{2}{3}\pi r^3$. It means that the volume of the sphere of radius r which is twice the volume of the hemispherical bowl of same radius is $2 \times \frac{2}{3}\pi r^3$ or $\frac{4}{3}\pi r^3$.

Remark : Another activity that may help us to obtain the formula for the volume of a hemisphere may be performed as follows :

Take a hemispherical bowl of radius r. Take also a conical vessel of base radius r and height r. Now fill up the conical vessel with water upto the brim and pour this water in the hemispherical bowl. The bowl will not be full. We fill the conical vessel upto the brim once again and pour the water in the bowl. We now observe that the bowl is full to the brim. From this, we get that

Capacity of the bowl = 2 times the capacity of the conical vessel

Since the capacity of the conical vessel is $\frac{1}{3}\pi r^3$ (the height of the conical vessel is r), therefore, the capacity of the hemispherical bowl of radius r is $2 \times \frac{1}{3}\pi r^3$ or $\frac{2}{3}\pi r^3$. This is the same result as we have obtained through Activity 5.

Example 8 : Find the volume of a spherical ball of radius 2.1 cm.

Solution : Here $r = 2.1$ cm

$$\therefore \quad \text{Volume } V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3} \times \frac{22}{7} \times 2.1 \times 2.1 \times 2.1 \text{ cm}^3$$

$$= 38.808 \text{ cm}^3$$

Example 9 : The capacity of a hemispherical tank is 155.232 l. Find its radius.

Solution : Suppose that the radius of the tank is r cm. Then its volume

$$V = \frac{2}{3}\pi r^3 \text{ cm}^3$$

Also, $155.232\ l = 155.232 \times 1000 \text{ cm}^3 = 155232 \text{ cm}^3$

$\therefore$ $$\frac{2}{3}\pi r^3 = 155232$$

or $$r^3 = \frac{155232 \times 3 \times 7}{2 \times 22}$$

$$= 3528 \times 3 \times 7$$

$$= (2 \times 3 \times 7)^3$$

$\therefore$ $$r = 2 \times 3 \times 7 = 42$$

Thus, the required radius is 42 cm.

Example 10 : A copper wire 36 m long with diameter 2 mm is melted to form a sphere. Find

(i) volume of the wire,
(ii) volume of the sphere,
(iii) the radius of the sphere.

Solution : (i) The radius of the wire $= \frac{1}{2} \times 2 \text{ mm} = 1 \text{ mm} = 0.1 \text{ cm}$ and

the length of the wire = 36 m = 3600 cm.

$\therefore$ Volume of the wire $= \pi \times (0.1)^2 \times 3600 \text{ cm}^3$

$$= 36\ \pi \text{ cm}^3$$

(ii) Volume of the sphere = Volume of the wire

$$= 36\ \pi \text{ cm}^3$$

(iii) Let the radius of the sphere be r cm. Then,

$$\frac{4}{3}\pi r^3 = 36\ \pi$$

i.e., $$r^3 = \frac{36 \times 3}{4}$$

or $$r = 3$$

Hence, the required radius is 3 cm.

Example 11 : A hemispherical bowl of steel is of thickness 0.3 cm (Fig. 15.5). If the inner radius of the bowl is 7 cm, find the volume of the steel used in making the bowl.

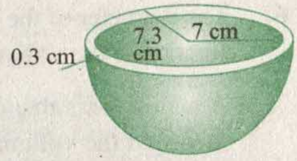


Fig. 15.5

Solution : The bowl may be looked upon as the portion trapped in between two hemispheres of radii 7 cm and 7.3 cm. Now

$$\text{Volume of the inner hemisphere} = \frac{2}{3} \times \pi \times 7^3 \text{ cm}^3$$

$$\text{Volume of the outer hemisphere} = \frac{2}{3} \times \pi \times (7.3)^3 \text{ cm}^3$$

$$\therefore \quad \text{Volume of the steel used} = \left[\frac{2}{3} \times \pi \times (7.3)^3 - \frac{2}{3} \times \pi \times 7^3\right] \text{cm}^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times \left[(7.3)^3 - 7^3\right] \text{cm}^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times 46.017 \text{ cm}^3$$

$$= 96.42 \text{ cm}^3$$

EXERCISE 15.3

1. Find the volume of a sphere whose radius is
   (i) 7 cm (ii) 3.5 dm (iii) 0.63 m
2. Find the volume of the water that a spherical solid ball of following diameter will replace:
   (i) 28 cm (ii) 0.21 m (iii) 3.5 dm
3. Express the volumes in Question 2 above in terms of litres.
4. How many litres of milk can a hemispherical bowl of diameter 10.5 cm hold?
5. The diameter of a metallic ball is 4.2 cm. What is the mass of the ball, if the density of the metal is 8.9 g per cm^3 ?

6. The diameter of the Moon is approximately one fourth the diameter of the Earth. What fraction is the volume of the Moon of the volume of the Earth?

7. A hemispherical tank is made up of an iron sheet 1 cm thick. If the inner radius is 1 m, then find the volume of the iron used to make the tank.

8. Find the volume of a sphere whose surface area is 154 cm^2.

9. A dome of a building is in the form of a hemisphere. From inside, it was white washed at the cost of Rs 498.96. If the rate of white wash is Rs 2.00 per square metre, find
   (i) inside surface area of the dome,
   (ii) the volume of the air inside the dome.

10. 27 solid iron spheres each of radius r and surface area S are melted to form a sphere with surface area S′. Find the
    (i) radius r' of the new sphere, (ii) ratio of S and S′.

Things to Remember

1. The magnitude of the region occupied by a solid in space is called its volume.

2. A standard unit of volume is cm^3 (or cu cm). Some other standard units are mm^3, dm^3, m^3 and km^3.

3. The capacity of a vessel is the volume of the liquid it can hold. The unit of capacity is litre (l).

4. $1\ l = 10^3\ cm^3$, $1\ ml = 1 cm^3$

5. Volume of a right circular cylinder:
$$V = (\text{Area of the base}) \times (\text{Height}) = \pi r^2 h$$

6. Volume of a right circular cone:
$$V = \frac{1}{3} \times (\text{Area of the base}) \times (\text{Height}) = \frac{1}{3}\pi r^2 h$$

7. Volume of a sphere:
$$V = \frac{4}{3} \times \pi \times (\text{radius})^3 = \frac{4}{3}\pi r^3$$

8. Volume of a hemisphere:
$$V = \frac{2}{3} \times \pi \times (\text{radius})^3 = \frac{2}{3}\pi r^3$$

As History Tells Us

On the basis of the clay tablets excavated from Nippur as also by reference to Ahme's Papyrus, we may say confidently that the Babylonians had the fundamental concepts of Geometry. It is clear from the tablets that as early as 1550 B.C., the Babylonians could find the areas of rectangles (including squares), right triangles and trapeziums. Quite likely, they had some conception of the area of a circle. They could also find the volumes of cuboids (and possibly of cylinders). In Ahme's papyrus, formula for the area of an isosceles triangle is given as $\frac{1}{2}bh$, the formula that we use today. The formula for the area of a circle of diameter d is given as $(d - \frac{1}{9}d)^2$, which leads to a value 3.1605 of π.

Whereas the Geometry used by Egyptians and Indians was *applied* Geometry, the Geometry developed by Greeks was *abstract*. Recall that Egyptians used Geometry basically for measuring areas and building pyramids and Indians used Geometry for constructing altars, planning the lay-out of colonies, and water systems, and constructing roads and buildings. But Greek Geometry was devoid of physical content. Thus, according to them, point had no dimension, viz. (Can you plot it?), line had one dimension length (Can you draw it without thickness or height?) and so on. Other abstract Geometries have since been developed. In the Greek Geometry, the present form of a postulate given by Euclid is, *given a line and a point not lying on the line, exactly one line can be drawn that goes through the given point and is parallel to the given line*. Geometries have been developed in one of which more than one line can be drawn through the given point and parallel to the given line, whereas in the other, it is possible that no line through the point may be parallel to the line. (Interestingly, physical models of both these Geometries exist).

Some of the names of ancient Indian mathematicians who contributed to Geometry are *Baudhayana* (800 B.C.) who gave a proof of a Theorem known as Pythagoras Theorem, *Brahamagupta* (born 598) who worked out the area of a (cyclic) quadrilateral in terms of its sides and the semi-perimeter, *Bhaskara* (born 1114) who furnished a (dissection) proof of Pythagoras theorem and *Aryabhata* (born 476) who worked out the area of an isosceles triangle, volume of a pyramid, a sphere, and also gave a value of π that comes out to be correct upto 4 decimal places.

Ancient people used various techniques of finding the value of π. *Archimedes* used the areas of two polygons of 96 sides, one inscribed in the circle and the

other circumscribing the circle from where he deduced that $3\frac{10}{71} \qquad 3\frac{1}{7}$, i.e., $3.1408 < \pi < 3.1428$. Many people used infinite series to denote the values of π. For example, *Nilakantha* (15th Century) and *Leibnitz* (1674) gave the value of

π as $\frac{\ }{4} \quad 1 \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{7} \quad \frac{1}{9}$.....

The all-time great Indian mathematician, Srinivas Ramanujan (1887-1920) wrote a paper *Modular Equations and Approximations to* π in 1914. In this paper, he gave fifteen different series to calculate $\frac{1}{\ }$. Based on some of these, D.H. Bailey of NASA and J.M.Borwein and P.B.Borwein of Simon Frazer University have developed computer algorithms by which value of π has been calculated to billions of digits.

CHAPTER

16

STATISTICS

16.1 Introduction

In Class VII, we have started the study of Statistics with the reading, interpretation and construction of pictographs and bar graphs (bar charts) representing any given information or data. You have seen how this pictorial representation of data enabled us to get some useful information about the given data at a glance. In this Chapter, we shall begin the study of Statistics in a formal way. We shall learn to find the arithmetic mean of raw data. We shall also learn to represent a given information or data into the form of an ungrouped or a grouped frequency distribution table. Reading and interpreting of *histograms* will also be discussed in this Chapter.

16.2 Raw Data

In our daily life, we often come across graphs and tables relating to some data in newspapers, magazines, information bulletins, etc. Let us see how these data are obtained.

Let us suppose that we are interested in knowing who, among the 20 students of Class VIII, secured the highest marks in Mathematics in the half-yearly examination and who secured the lowest marks. We may be interested in many more things, for instance, how many students secured 60 or more marks and how many students secured less than 33 marks, and so on. How do we obtain the desired information? We may ask each of the 20 students his or her marks in Mathematics in the half-yearly examination. Suppose that these 20 students obtained the following marks (out of 100) :

45, 56, 61, 31, 56, 33, 70, 61, 76, 36, 56, 59, 64, 56, 88, 28, 56, 70, 64, 74

The information received above from each student, i.e., the marks obtained by each student, is called an *observation*. Such observations gathered initially are called *raw data*.

We may list the names of all the 20 students and show their marks, starting from the lowest, against their names. Thus, we organise the above raw data in the form of a table as shown below :

Table 16.1 : Marks obtained by 20 Students in Mathematics

Name	Marks
Deepak	28
Rehana	31
Arun	33
Sonal	36
Neetu	45
Raghu	56
William	56
Prakash	56
Ravi	56
Sohan	56
Urmil	59
Nelson	61
Haider	61
Sudha	64
Tina	64
Gorang	70
Malleshwari	70
Subramaniam	74
Abdul	76
Mary	88

Let us now look at Table 16.1. What does it reveal? A glance at the table shows that the highest marks obtained are 88 (obtained by Mary), whereas the lowest are 28 (obtained by Deepak).

The difference between the highest and the lowest values of the observations is called the range of the data (or observations).

Thus, the range of the above data is (88 – 28) marks, i.e., 60 marks. We also wanted to know how many students secured 60 or more marks and how many students secured less than 33 marks. We can count the number of such students in the table and find out that 9 out of 20 students obtained 60 or more marks and 2 out of 20 students obtained less than 33 marks.

Remark : Tables are written horizontally also. For example,

Name	Kamla	Salma	Kavita	Shashi
Weight (in kg)	50	55	53	48

16.3 Arithmetic Mean

You must have heard people talking about the average height, average speed, average weight, and average score (marks), etc. What do we mean by the word *average*? An average is a number indicating the representative or central value of a group of observations or data. If we are told that the average height of Class VIII students is 149 cm, we may think that, in general, the heights of the students are spread around 149 cm. We know that not all students in the Class have 149 cm height, some have a lower height and some have a higher height. But knowledge of average height gives us a general impression of the heights of the students in the Class as a whole.

Averages are of different types. Here, we shall learn about the simplest type of average commonly known as the *arithmetic mean* or simply the *mean*. It is calculated by dividing the sum of all the observations by the total number of observations. Thus, if M is the mean of the given observations, then

$$\text{Mean} = \text{M} = \frac{\text{Sum of all observations}}{\text{Total number of observations}}$$

For the example given in Section 16.2 relating to marks of 20 students,

$$\text{M} = \frac{\begin{array}{c}28 + 31 + 33 + 36 + 45 + 56 + 56 + 56 + 56 + 56 + 59 + \\ 61 + 61 + 64 + 64 + 70 + 70 + 74 + 76 + 88\end{array}}{20} \text{ marks}$$

$$= \frac{1140}{20} \text{ marks} = 57 \text{ marks}$$

Thus, the required mean is 57 marks.

Remark : It should be noted that the mean in the above example is 57, which is different from all the observations in the raw data. However, it is possible for the mean to be one of the observations.

Let us consider some examples to illustrate the same.

Example 1 : Following are the ages (in years) of 10 teachers in a school :

32, 41, 28, 54, 35, 26, 23, 33, 38, 40

(i) What is the age of the oldest teacher and that of the youngest teacher?
(ii) What is the range of the ages of the teachers?
(iii) What is the mean age of these teachers?

Solution : (i) Arranging the ages in ascending order (23, 26, 28, 32, 33, 35, 38, 40, 41, 54), we find that the age of the oldest teacher in the school is 54 years and that of the youngest teacher is 23 years.

(ii) The range is (54 – 23) years, i.e., 31 years.

(iii) $$M = \frac{\text{Sum of all observations}}{\text{Total number of observations}}$$

$$= \frac{32+41+28+54+35+26+23+33+38+40}{10} \text{ years}$$

$$= \frac{350}{10} \text{ years} = 35 \text{ years}$$

The mean age of the teachers is 35 years. Note that, in this case, mean is one of the observations of the raw data.

Example 2 : Following are the weights (in kg) of 8 students of a Class :

48.5, 50, 44.5, 49.5, 50.5, 45, 51, 43

(i) Find the mean weight.
(ii) What will be the mean weight if a teacher, whose weight is 62 kg, is also included?

Solution : (i) $$M = \frac{48.5+50.0+44.5+49.5+50.5+45.0+51.0+43.0}{8} \text{ kg}$$

$$= \frac{382.0}{8} \text{ kg} = 47.75 \text{ kg} = 47.8 \text{ kg correct to one decimal place.}$$

Thus, the mean weight is 47.8 kg.

(ii) If weight of the teacher is also included, then sum of all the observations (in kg)

$$= 48.5 + 50.0 + 44.5 + 49.5 + 50.5 + 45.0 + 51.0 + 43.0 + 62.0$$

$$= 444.0$$

$\therefore$ Number of observations = 8 + 1 = 9

Hence, mean $= \frac{444}{9}$ kg $= 49\frac{1}{3}$ kg = 49.3 kg correct to one decimal place.

Hence, the new mean weight will be 49.3 kg.

Example 3 : The mean of 6 observations was found to be 40. Later on, it was detected that one observation 82 was misread as 28. Find the correct mean.

Solution : Mean of 6 observations = 40

$\therefore$ Sum of all the observations $= 40 \times 6 = 240$

In the above observations, 82 was misread as 28.

Therefore, correct sum of all the observations $= 240 - 28 + 82 = 294$

Hence, correct mean $= \frac{294}{6} = 49$

Example 4 : Mean temperature of a certain week was 25°C. If mean temperature of Monday, Tuesday, Wednesday and Thursday was 23°C and that of Thursday, Friday, Saturday and Sunday was 28°C, find the temperature of Thursday.

Solution : Mean temperature of the week = 25 °C

$\therefore$ Sum of the temperatures of 7 days $= 7 \times 25\,°C = 175\,°C$ (1)

Sum of the temperatures of Monday, Tuesday, Wednesday and Thursday

$= 4 \times 23°C = 92\,°C$ (2)

Sum of the temperatures of Thursday, Friday, Saturday and Sunday

$= 4 \times 28°C = 112\,°C$ (3)

$\therefore$ Sum of the temperatures of Monday to Sunday and Thursday

$= 92\,°C + 112°C$ [From (2) and (3)]

$= 204°C$ (4)

$\therefore$ Temperature of Thursday $= 204°C - 175°C$ [From (1) and (4)]

$= 29°C$

16.4 Frequency Distribution Table

Let us again refer to the example of marks of 20 students of Class VIII in Mathematics (Table 16.1). It is quite possible that in many situations, two or more observations are identical. Do you see that each of the students Raghu, William, Prakash, Ravi and Sohan has obtained 56 marks? In other words, 56 marks were obtained by 5 students. Similarly, 61 marks were obtained by 2 students. At times we are interested in knowing which observation occurs how often.

The number of times a particular observation occurs is called its frequency.

Having known that 5 students obtained 56 marks, we say that the frequency of 56 is 5. Similarly, the frequency of 61 is 2.

If we do this for all the observations and rearrange the data given in Table 16.1, starting from the lowest to the highest observation (or from highest to the lowest observation), we will obtain the *frequency distribution* of the marks of 20 students. A table in which such a distribution of frequencies is given is called a *frequency distribution table* or simply a *frequency table*. We thus obtain the following frequency distribution table of the marks of 20 students :

Table 16.2 : Frequency Distribution of Marks of 20 Students of Class VIII in Mathematics

Marks Obtained	*Frequency*
28	1
31	1
33	1
36	1
45	1
56	5
59	1
61	2
64	2
70	2
74	1
76	1
88	1
Total	**20**

Do you observe that Table 16.2 is more meaningful as compared to Table 16.1? It is so because it conveys certain important features of the data at a glance. For example, it shows at once that 5 students have obtained 56 marks.

16.5 Use of Tally Marks

In the above example, the number of observations was 20 only and therefore, it was convenient to count and find the respective frequencies from the given raw data. However, when the number of observations is large, it may not be convenient to find the frequencies by simple counting. In such cases, we make use of bars (|,\) called *tally marks* which are quite helpful in finding the frequencies. For ease in counting, tallies are usually marked in *bunches of five*. The first four tallies are marked vertically. The fifth tally in a bunch is marked diagonally across the earlier four (~~||||~~). We illustrate the method of marking tallies through an example given below :

Example 5 : Heights (in cm) of 30 girls in a Class are given below :

140, 140, 160, 139, 153, 153, 146, 150, 148, 150, 152, 146, 154, 150, 160, 148, 150, 148, 140, 148, 153, 138, 152, 150, 148, 138, 152, 140, 146, 148

Prepare a frequency distribution table for the above data.

Solution : We make a table with three columns with headings *Height (in cm)*, *Tally marks* and *Frequency* as shown in Table 16.3.

Table 16.3: Frequency Distribution of Heights of 30 Girls

Height (in cm)	*Tally Marks*	*Frequency*
138	||	2
139	|	1
140	||||	4
146	|||	3
148	~~||||~~ |	6
150	~~||||~~	5
152	|||	3
153	|||	3
154	|	1
160	||	2
	Total	**30**

In the first column entitled *Height* (*in cm*), we write different heights in ascending order as 138, 139, 140, ... We now look at the given data. The first observation is 140. So we place a tally mark against 140 in the *Tally Marks* column. The second observation is again 140. So we again place a tally mark against 140 in the Table. The third observation is 160. So we place a tally mark against 160 in the Table, and so on. When all the observations are exhausted, we count the tally marks against each height and enter the corresponding number in the *frequency* column. We thus obtain the following table :

Remark : In both the frequency distribution tables (Table 16.2 and Table 16.3), we note that *sum of all the frequencies is equal to the total number of observations.*

EXERCISE 16.1

1. Find the arithmetic mean of the scores 8, 6, 10, 12, 1, 3, 4, 4.
Also, find the range of the data.

2. The enrolment of a school during six consecutive years was as follows :

1620, 2060, 2540, 3250, 3500, 3710

Find the mean enrolment of the school for this period.

3. The marks (out of 100) obtained by a group of students in a Science test are 81, 72, 90, 90, 86, 85, 92, 70 and 71. Find the mean marks obtained by the group.

4. The heights of 10 girls were measured in cm and the results were as follows :

143, 148, 135, 150, 128, 139, 149, 146, 151, 132

(i) What is the height of the tallest girl?

(ii) What is the height of the shortest girl?

(iii) What is the range of the data?

(iv) Find the mean height.

(v) How many girls are there whose heights are less than the mean height?

5. The rainfall (in mm) in a city on 7 days of a certain week were recorded as follows :

Day	Mon.	Tue.	Wed.	Thu.	Fri.	Sat.	Sun.
Rainfall (in mm)	2.2	21.3	25.6	0.0	4.9	0.0	5.5

(i) Find the range of the rainfall in the above data.

(ii) Find the mean rainfall for the week.

(iii) On how many days was the rainfall less than the mean rainfall?

6. The maximum daily temperatures (in °C) of a city on 10 consecutive days are given below :

32.4, 29.5, 26.3, 25.7, 23.4, 24.2, 22.4, 22.5, 22.8, 23.3

(i) Find the range of the data. (ii) Find the mean daily temperature.

7. The numbers of babies born in a hospital on the first seven days of March, 2003 are : 18, 18, 17, 17, 10, 11, 14. Find the mean number of babies born in the hospital during the above seven days.

8. A group of students was given a special test. The test was completed by the various students in the following time (in minutes) :

17, 19, 20, 22, 24, 24, 28, 30, 30, 36

(i) Find the mean time taken by the students to complete the test.

(ii) How many students took more than the mean time to complete the test?

(iii) If the student who took 36 minutes had taken only 22 minutes to complete the test, what would have been the mean time?

9. The mean of 5 numbers is 20. If one number is excluded, mean of the remaining numbers becomes 23. Find the excluded number.

10. The mean of 25 observations is 27. If one observation is included, the mean still remains 27. Find the included observation.

11. The mean of 5 observations is 15. If mean of the first three observations is 14 and that of the last three is 17, find the third observation.

12. Find the mean of the first ten natural numbers.

13. Find the mean of the first six prime numbers.

14. The runs scored by a player in nine cricket matches are 85, 82, 91, 0, 42, 8, 29, 1 and 37. Find :

(i) mean (average) runs scored by the player.

(ii) range of the runs scored.

15. Mean of 9 observations was found to be 35. Later on, it was detected that an observation 81 was misread as 18. Find the correct mean of the observations.

16. The scores (out of 100) obtained by 33 students in a Mathematics test are : 69, 48, 84, 58, 84, 48, 73, 83, 48, 66, 58, 66, 64, 71, 64, 66, 69, 66, 83, 66, 69, 71, 81, 71, 73, 69, 66, 66, 64, 58, 64, 69, 69.

Prepare a frequency table for the above scores.

17. The number of members in 20 families of a township are 6, 8, 4, 3, 5, 6, 7, 4, 3, 4, 5, 6, 4, 5, 4, 3, 3, 6, 4 and 3. Prepare a frequency distribution table for the data and answer the following questions :

(i) What is the smallest family size? How many families are of this size?

(ii) What is the largest family size? How many families are of this size?

(iii) What is the most common family size?

18. In a study of number of road accidents in a city, the observations for the 30 days of April, 2003 were 4, 3, 5, 6, 4, 3, 2, 5, 4, 2, 6, 2, 1, 2, 2, 0, 5, 4, 6, 1, 3, 0, 5, 3, 6, 1, 5, 5, 2 and 6. Prepare a frequency distribution table for the above data.

19. A die is a cube where six faces are marked with numbers (or dots) from 1 to 6 (one number on one face), [Fig. 16.1]. The scores obtained in 25 throws are 5, 4, 3, 2, 1, 1, 2, 5, 4, 6, 6, 6, 3, 2, 1, 4, 3, 2, 1, 5, 6, 5, 2, 1 and 3. Prepare a frequency table for the above scores.

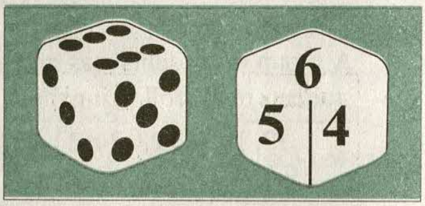


Fig. 16.1

20. The weights (in kg) of 30 students of a Class are 49, 48, 46, 48, 48, 50, 52, 53, 54, 43, 43, 41, 43, 55, 56, 48, 47, 41, 40, 49, 51, 51, 56, 46, 45, 44, 49, 53, 42 and 47. Prepare a frequency table for the above data and answer the following questions :

(i) What is the least weight?

(ii) Find the number of students having the least weight in the above data.

(iii) Find the number of students having the maximum weight in the above data.

(iv) Which weight is possessed by the maximum number of students?

21. Construct a frequency distribution table for the following marks (out of 50) obtained by 60 students in Social Science and find the range :

23, 24, 27, 37, 40, 41, 42, 21, 17, 19, 34, 36, 29, 38, 42, 39, 42, 36, 24, 21, 24, 19, 18, 27, 25, 41, 40, 30, 29, 27, 31, 37, 34, 36, 37, 42, 36, 35, 43, 37, 29, 24, 25, 29, 28, 46, 29, 39, 41, 32, 31, 32, 17, 18, 19, 23, 24, 42, 36, 35

22. The value of up to 50 decimal places is given below:

3. 1415926535 8979323846 2643383279 5028841971 6939937510

Write the frequencies of the following digits in the decimal part of the above number:

(i) 2 (ii) 3 (iii) 5 (iv) 6 (v) 9 (vi) 1

16.6 Grouping of Data

We have learnt how to organise data in the form of a table in terms of frequencies. Such a table is called an *ungrouped frequency distribution table* of raw data. In our first example of marks, we had data of only 20 students. If the number of different observations is quite large (e.g. Question 21 of Exercise 16.1), it is desirable to condense the given data into several groups and obtain a frequency distribution of the number of observations falling in each group. When the data are written in this form, the data are said to be

grouped and the distribution obtained is called a *grouped frequency distribution*. We illustrate this concept below.

Table 16.4 : Grouped Frequency Distribution of Marks of 60 Students in Social Science

Class Interval (Marks)	*Tally Marks*	*Frequency*
17 – 19	~~IIII~~ II	7
20 – 22	II	2
23 – 25	~~IIII~~ IIII	9
26 – 28	IIII	4
29 – 31	~~IIII~~ III	8
32 – 34	IIII	4
35 – 37	~~IIII~~ ~~IIII~~ I	11
38 – 40	~~IIII~~	5
41 – 43	~~IIII~~ IIII	9
44 – 46	I	1
	Total	**60**

In the above table, we have condensed the given 60 observations into ten groups 17–19, 20 – 22, . . . , 44 – 46. Each of these groups is called a *Class Interval* (or briefly a *Class*).

We can also group the same 60 observations into ten groups 17– 20, 20 – 23, 23 – 26, . . ., and 44 – 47 as shown in Table 16.5.

It appears from the classes 17 – 20, 20 – 23, etc. of Table 16.5 that the observation 20 may belong to both the classes 17 – 20 and 20 – 23. Similarly, observation 23 may belong to both the classes 20 – 23 and 23 – 26, and so on. But no observation could belong simultaneously to two classes. To avoid this difficulty, we adopt the convention that the common observation 20 belongs to the higher class, i.e., 20 – 23 (and not to 17 – 20). Similarly, 23 belongs to 23 – 26 (and not to 20 – 23), and so on. For example, class 26 – 29 contains all observations which are greater than or equal to 26 but less than 29, etc. In this Chapter, we shall take groups or class intervals as given in Table 16.5.

In the class interval 17 – 20, 17 is called the *lower class limit* and 20 is called the *upper class limit*. Similarly, for class interval 41 – 44, 41 is the lower class limit and 44

Table 16.5 : Grouped Frequency Distribution of Marks of 60 in Students Social Science

Class Interval (Marks)	*Tally Marks*	*Frequency*
17–20	卌 ||	7
20–23	||	2
23–26	卌 ||||	9
26–29	||||	4
29–32	卌 |||	8
32–35	||||	4
35–38	卌 卌 |	11
38–41	卌	5
41–44	卌 ||||	9
44–47	|	1
	Total	**60**

is the upper class limit. The difference between the upper and the lower class limits of an interval is called the *width* or *size* of the interval. Clearly, in the above distribution, *width* or size of each interval is 3. The frequency corresponding to a class interval is called its *class frequency*. Further, the mid-point of a class is called its *class mark*. It is obtained by taking the mean of lower and upper class limits. For example, class mark of the interval 17 – 20 in Table 16.5 is $\frac{17+20}{2} = 18.5$. Similarly, class marks of the other intervals are 21.5, 24.5, and so on.

Remarks 1 : There is no hard and fast rule for choosing the number and size of class intervals in a grouped frequency distribution. The size of each group and the number of groups are decided, keeping in view the range of the data.

2. The class sizes (widths) of different classes may not necessarily be the same. However, discussion of such class sizes is beyond the scope of this book.

Let us take some examples to illustrate these concepts :

Example 6 : Study the following frequency distribution table (Table 16.6) and answer the questions given below :

(i) What is the size of the class intervals?
(ii) What is the lower class limit of the second class interval?
(iii) What is the upper class limit of the seventh class interval?
(iv) Which class has the highest frequency?
(v) What is the class mark of the fourth class interval?

Table 16.6 : Frequency Distribution of Daily Income of 600 Workers of a Factory

Class Interval (*Daily Income in Rupees*)	*Frequency* (*Number of Workers*)
100–125	45
125–150	25
150–175	55
175–200	50
200–225	125
225–250	140
250–275	55
275–300	35
300–325	50
325–350	20
Total	**600**

Solution : (i) Size of each class interval is 25.

(ii) Lower class limit of the second class interval (125–150) is 125.

(iii) Upper class limit of the seventh class interval (250–275) is 275.

(iv) Class 225 – 250 has the highest frequency (140).

(v) Class mark of the fourth class interval (175–200) is

$$\frac{175+200}{2}, \text{i.e.}, \frac{375}{2}, \text{i.e.}, 187.5$$

Example 7 : Population (in hundreds) of 80 towns and villages of a State, taken at random from a Census Report are 11, 72, 15, 8, 15, 3, 23, 26, 2, 319, 200, 6, 16, 6, 131, 5, 18, 240, 99, 127, 31, 72, 18, 30, 43, 2, 1, 52, 40, 3, 7, 13, 5, 142, 70, 86, 31, 38, 70, 51, 11, 52, 18, 46, 89, 1, 30, 25, 4, 52, 15, 139, 12, 277, 24, 48, 5, 26, 39, 18, 17, 159, 30, 171, 30, 6, 160, 52, 222, 13, 55, 9, 3, 149, 3, 52, 12, 124, 120 and 10.

Prepare a grouped frequency table for the above data using class intervals 0 – 30, 30 – 60, 60 – 90, and so on. Which group has the highest frequency?

Solution : We have the following table :

Table 16.7 : Frequency Distribution of Population (in hundreds) of 80 Towns and Villages of a State

Population (in hundreds)	*Tally Marks*	*Frequency*
0 – 30	𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 𝍸 ||||	39
30 – 60	𝍸 𝍸 𝍸 ||||	19
60 – 90	𝍸 |	6
90 – 120	|	1
120 – 150	𝍸 ||	7
150 – 180	|||	3
180 – 210	|	1
210 – 240	|	1
240 – 270	|	1
270 – 300	|	1
300 – 330	|	1
	Total	**80**

Group 0 – 30 has the highest frequency.

16.7 Histograms

In Class VII, you have learnt how to represent given data by means of a bar graph. Let us now see how a grouped frequency distribution can be represented graphically. The most common graphical representation used for the same is *histogram* (Fig. 16.2).

To make a histogram, we draw two perpendicular axes and choose a suitable scale for each axis. We mark class intervals of the grouped data on the horizontal axis and the respective class frequencies on the vertical axis. For each class, a rectangle is constructed with class interval as the base and height determined from the class frequency so that areas of the rectangles are proportional to the frequencies of their classes. As we are considering a grouped frequency distribution of class intervals with equal width only, the heights of the rectangles will be proportional to the respective frequencies.

Fig. 16.2 shows the histogram for the grouped frequency distribution of daily income of 600 workers of a factory (Table 16.6).

Histogram for the data of Table 16.6

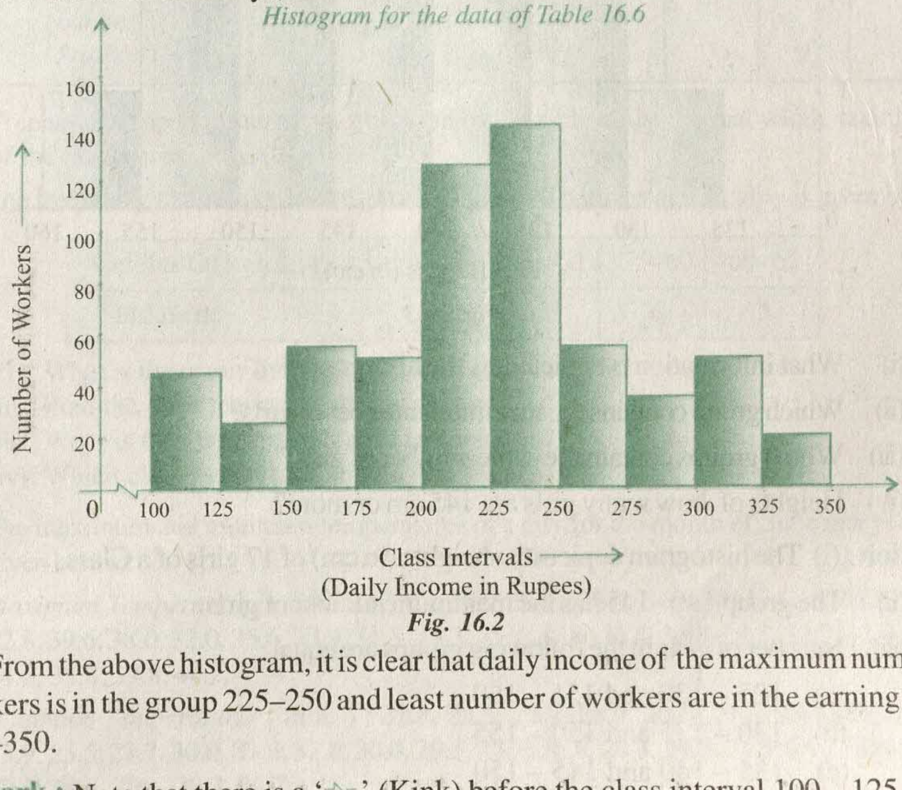


Fig. 16.2

From the above histogram, it is clear that daily income of the maximum number of workers is in the group 225–250 and least number of workers are in the earning group 325–350.

Remark : Note that there is a '⌇' (Kink) before the class interval 100 – 125 on the horizontal axis. It shows that the full distance 0 to 100 is not shown.

Example 8 : Read the following histogram and answer the questions given at the end :

Histogram for the Heights (in cm) of 17 Girls of a Class

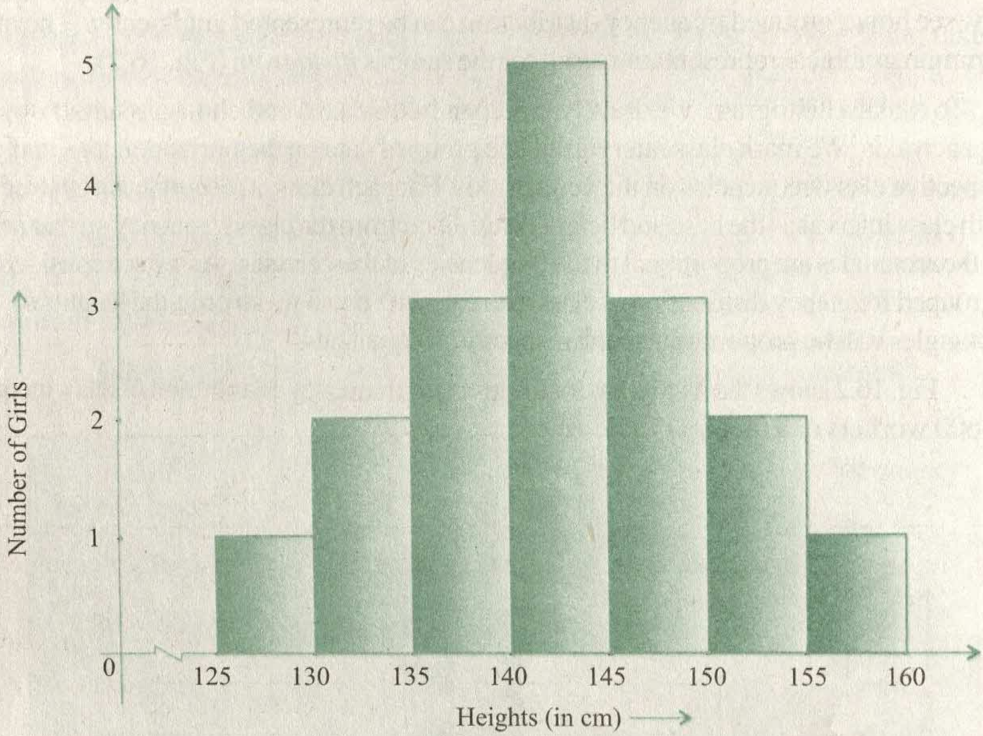


Fig. 16.3

(i) What information is depicted by the histogram?

(ii) Which group contains the maximum number of girls?

(iii) Which groups contain the same number of girls?

(iv) Heights of how many girls are 145 cm or more?

Solution : (i) The histogram depicts the heights (in cm) of 17 girls of a Class.

(ii) The group 140 – 145 has the maximum number of girls.

(iii) Number of girls in the following groups are equal :

(a) 125 – 130 and 155 – 160

(b) 130 – 135 and 150 – 155

(c) 135 – 140 and 145 – 150

(iv) Heights of 6 girls are 145 cm or more.

EXERCISE 16.2

1. The marks obtained in Mathematics by 40 students of a Class in an examination are 3, 20, 13, 1, 21, 13, 3, 23, 16, 13, 5, 24, 15, 7, 10, 18, 18, 7, 17, 21, 15, 5, 23, 2, 12, 20, 2, 10, 16, 23, 18, 12, 6, 9, 7, 3, 5, 16, 8 and 8.

   Present the data in the form of a grouped frequency distribution, using class intervals of equal size, one of the class intervals being 10 – 15.

2. Weekly savings (in rupees) of 30 students of Class VIII are 38, 42, 40, 35, 72, 27, 57, 62, 59, 80, 84, 73, 65, 40, 76, 40, 38, 60, 58, 38, 54, 39, 50, 44, 71, 83, 45, 38, 80 and 77.

   Construct a grouped frequency table with class intervals of equal width such as 30 – 35.

3. The following data give the pocket expenses of 100 students of a school:

Weekly Pocket Expenses (in rupees)	30	35	45	50	55	60	65
Number of Students	6	10	14	22	35	9	4

   Prepare a grouped frequency distribution of class intervals of equal width, taking one of the class intervals as 30 – 40.

4. The frequency distribution of weights (in kg) of 40 persons of a locality is given below.

Weights (in kg)	40–45	45–50	50–55	55–60	60–65
Frequency	4	12	13	6	5

   (i) What is the upper limit of the fourth class interval?
   (ii) Find the class marks of all the classes.
   (iii) What is the class size of each class interval?
   (iv) Which class interval has the highest frequency?

5. The maximum and minimum temperatures of a city for the month of June in a year are given below :

   Maximum Temperatures (in°C) : 35.5, 35.9, 36.0, 38.4, 36.6, 40.1, 41.3, 43.3, 42.8, 32.8, 39.6, 38.0, 32.0, 35.6, 33.9, 34.5, 35.3, 35.7, 35.9, 36.4, 33.8, 33.5, 32.7, 32.9, 34.0, 34.6, 38.8, 39.8, 40.2, 41.4.

   Minimum Temperatures (in°C) : 27.8, 23.4, 23.4, 28.0, 26.6, 29.5, 28.7, 33.5, 22.6, 23.9, 25.5, 21.7, 30.0, 31.3, 32.6, 30.0, 29.5, 25.5, 26.3, 24.3, 24.0, 23.5, 23.2, 30.6, 27.5, 28.3, 28.7, 29.6, 30.3, 32.7.

   Construct a grouped frequency table of each of the above, using equal class sizes,

taking one of the class intervals as 36 – 37 for the maximum temperatures and one of class intervals as 24 – 25 for the minimum temperatures.

6. The earnings of (approximated to nearest hundreds) 30 medical stores on a particular day are as follows :

Earnings (in rupees)	600	1500	1800	1900	2400	2600	3100	3900
Number of Stores	3	7	4	5	4	3	2	2

Prepare a grouped frequency distribution table taking class intervals of equal size, one such interval being 500 – 1000.

7. Pulse rate (per minute) of 30 persons were recorded as 61, 76, 72, 73, 71, 66, 78, 73, 68, 81, 78, 63, 72, 75, 80, 68, 75, 62, 71, 81, 73, 60, 79, 72, 73, 74, 71, 64, 76 and 71.

Construct a frequency table using class intervals of equal width, one class interval being 60 – 65.

8. Read the following histogram and answer the questions given at the end :

Histogram for the Marks Obtained by 43 Students of a Class in Physics

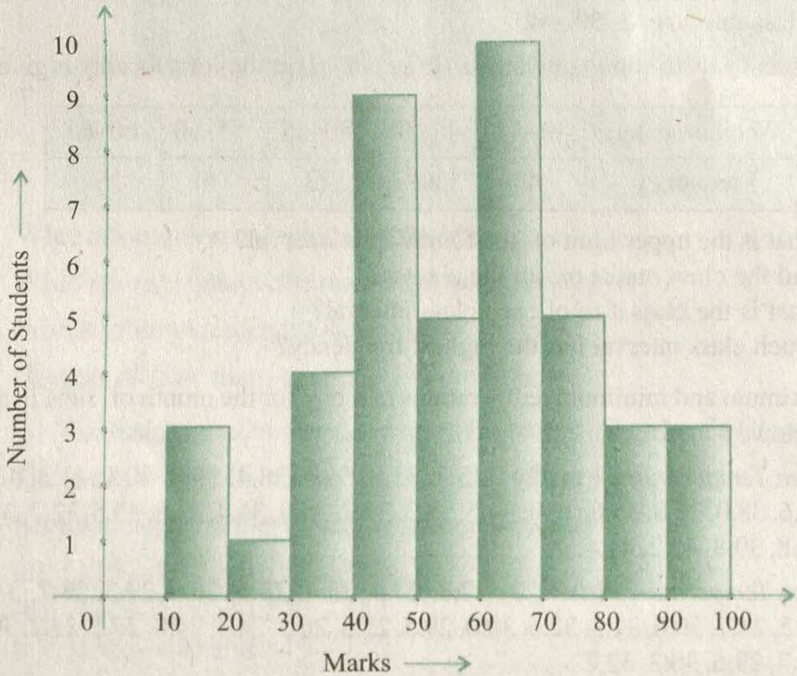


Fig. 16.4

(i) What information is depicted by the above histogram?
(ii) What is the size of each class?
(iii) Write the number of students in the highest marks group.
(iv) In which groups the number of students are the same?
(v) What is the number of students in the lowest marks group?
(vi) How many students have secured 60 or more marks?

9. Study the histogram given below and answer the questions following the histogram :

Histogram for the ages of 26 teachers of a School

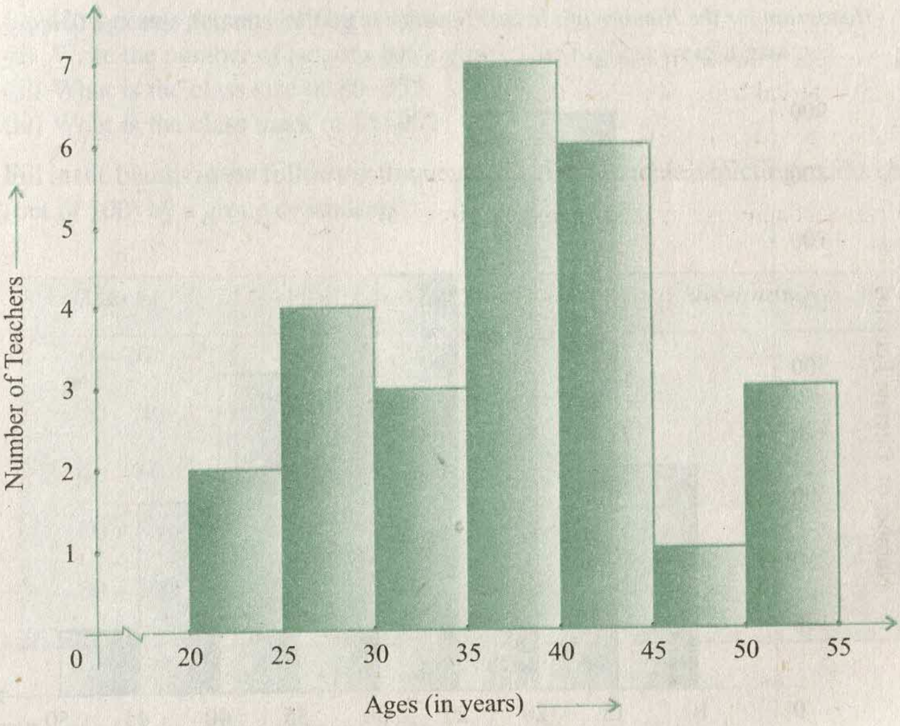


Fig. 16.5

(i) What information is depicted by the above histogram?
(ii) What is the number of teachers in the oldest age group in the school?
(iii) What is the number of teachers in the youngest age group in the school?
(iv) In which age group, the number of teachers is the least?
(v) In which age group, the number of teachers is the maximum?
(vi) What is the class size of each class interval?
(vii) What are the class marks of all the class intervals?
(viii) How many teachers are below 30 years in age ?

10. Construct a grouped frequency table with class intervals 0–5, 5–10 and so on for the following marks obtained in Biology (out of 50) by a group of 35 students in an examination:

0, 5, 6, 7, 10, 12, 14, 15, 20, 22, 25, 26, 27, 8, 11, 17, 3, 6, 9, 17, 19, 21, 22, 29, 31, 35, 37, 40, 42, 45, 49, 4, 50, 16, 20.

(i) What is the range of the data?

(ii) Which group contains the maximum number of students?

11. Read the following histogram and answer the questions :

Histogram for the Number of Literate Females of a Village in different age Groups

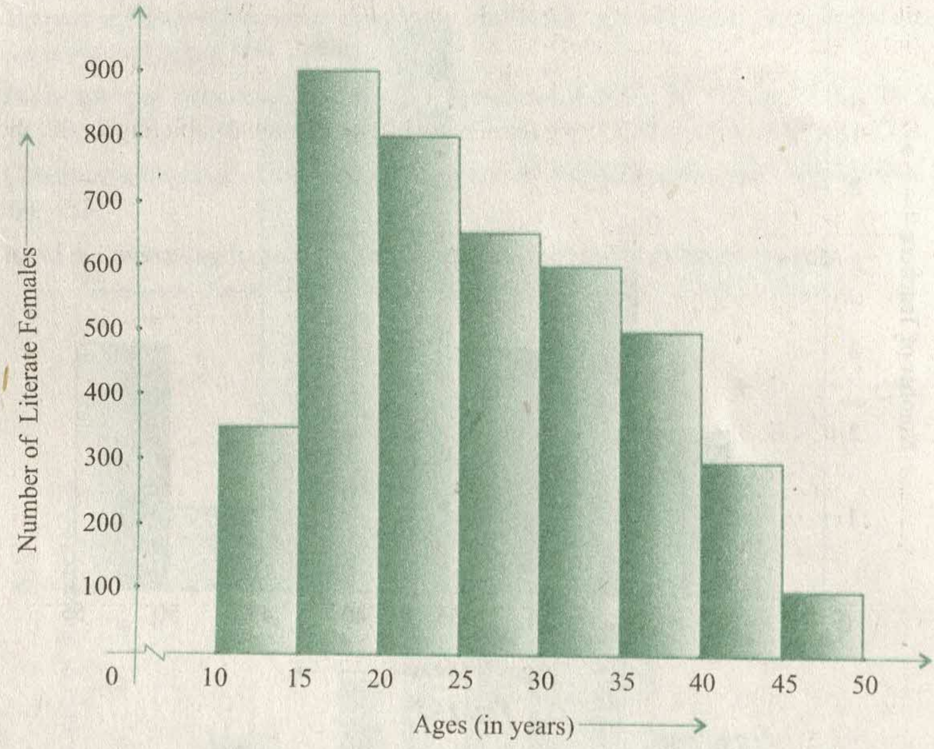


Fig. 16.6

(i) What information is depicted by the above histogram?

(ii) In which age group is the number of literate females maximum?

(iii) In which age group is the number of literate females minimum?

(iv) What is the width of each class interval?

(v) What are the class marks of all the class intervals?

(vi) How many females below 30 years in age are literate?

12. Weights (in kg) of 50 persons are given in the following grouped frequency distribution table :

Weights (in kg)	50–55	55–60	60–65	65–70	70–75	75–80	80–85	85–90
Number of Persons	12	8	5	4	5	6	6	4

(i) How many persons belong to group 50–55?
(ii) Write the number of persons belonging to the highest weight group.
(iii) What is the class size of 80–85?
(iv) What is the class mark of 85–90?

13. Fill in the blanks in the following frequency distribution table depicting marks obtained (out of 100) by a group of students :

Marks	*Tally Marks*	*Frequency*
0 – 20	III	—
20 – 40	卌 II	—
40 – 60	卌 卌 卌 —	18
60 – 80	卌 卌 I	—
80 – 100	—	2

14. Fill in the blanks in the following table :

Weights in (kg)	40–50	50–60	60–70	70–80	80–90
Class Marks	—	—	—	—	—

15. Prepare a grouped frequency table for the histogram given below:

Histogram for the Monthly Salaries of Employees of a Factory

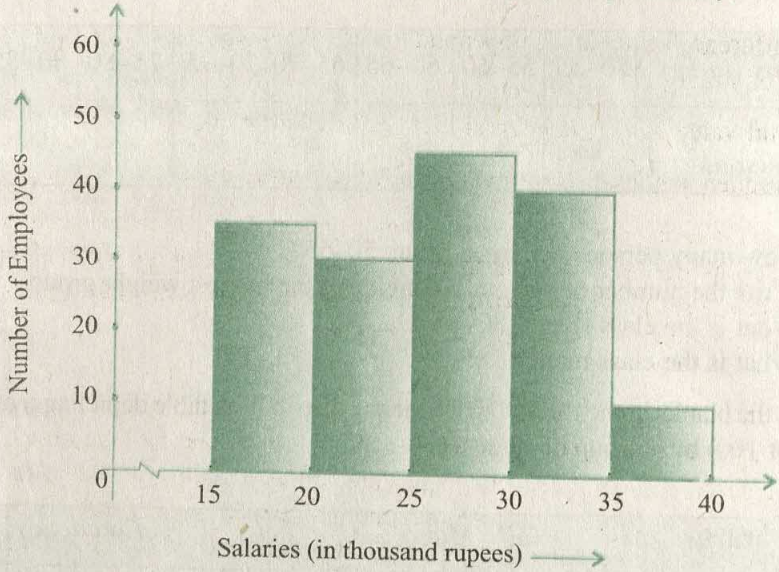


Fig. 16.7

Things to Remember

1. Observations gathered initially are called raw data.
2. The difference between the highest and the lowest values of the observations in given data is called the range.
3. Mean of given data = $\dfrac{\text{Sum of all observations}}{\text{Total number of observations}}$
4. In given data, the number of times a particular observation occurs is called its frequency.
5. A table showing the frequencies of various observations of data is called a frequency distribution table or simply a frequency table.
6. When the number of observations is large, we make use of tally marks to count the frequencies.
7. Tallies are usually marked in bunches of five for ease of counting.
8. When the list of observations is very large, we usually organise the data into groups called classes or class intervals. The data so obtained is called grouped data or a grouped frequency distribution.

9. The lower value of a class interval is called its lower class limit and the upper value of a class interval is called its upper class limit.

10. The difference between the upper and lower values of a class interval is called its width or size.

11. The mid-value of a class interval is called its class mark.

12. The frequency of a class interval is called its class frequency.

13. A histogram is a graphical representation of grouped data in which class intervals are taken along the horizontal axis and frequencies along the vertical axis. For each class, a rectangle is constructed with class interval as the base and its height is determined by the corresponding frequency.

As History Tells Us

All of us have some concept or the other of Statistics because all the magazines, newspapers, radio and TV advertisements are full of statistics or numerical data. Existence of the practice of collecting numerical data in ancient India is evident from the fact that during the reign of *Chandragupta Maurya* (324 B.C.–300 B.C.), there was a very good system of collecting such data specially in regard to the births and deaths. During *Akbar*'s reign (1556 – 1605), *Raja Todarmal*, the then Land and Revenue Minister, maintained good records of land and agricultural statistics. In *Ain-i-Akbari* written by *Abul Fazal* (in 1596 – 97), a detailed account of the administrative and statistical surveys conducted during that period can be found.

The first real Statistician in the modern sense is supposed to be an English shopkeeper *John Graunt* (1620 – 1674). He got interested in *jawfaln*, *King's–Evil*, *Planet* and *tissick* (all of these are diseases !) and other factors leading to death in London at that time. He collected data and processed the same. This was the beginning of Statistics.

Attracted by Graunt's technique, many mathematicians, *Laplace* (1749 – 1827) and *Gauss* (1777 – 1855) among them, developed the fundamental ideas of Statistics. Biologist caught on the idea and, using statistical techniques, established important theories in their field. Famous among them are *Charles Darwin* (1809-1882) and his theory of evolution, *Gregor Mendel* (1822–1884) and his peas-experiment, *Karl Pearson* (1857–1936) and his correlation theory that tells us how to determine whether or not one factor affects another and to what degree, if at all !

Today there is no arena of life where the subject of Statistics does not play an important role.

ANSWERS

EXERCISE 1.1

1. Numbers ending in 2,3,7,8 are not perfect squares.

2. (i) 1 (ii) 4 (iii) 1 (iv) 9 (v) 6
   (vi) 9 (vii) 4 (viii) 0 (ix) 6 (x) 5

3. (i) Number ending in odd number of zeros are not square numbers.
   (ii) Number ending in 2 are not square numbers.
   (iii) Same as (i)
   (iv) Same as (i)

4. (i) and (iii)

6. $100001^2 = 1\underline{0000}2\underline{0000}1$
   $10000001^2 = \underline{100000020000001}$

7. $1010101^2 = \underline{1020304030201}$
   $\underline{101010101}^2 = 10203040504030201$

8. $4^2 + 5^2 + \underline{20}^2 = 21^2$, $5^2 + \underline{6}^2 + 30^2 = 31^2$, $6^2 + 7^2 + \underline{42}^2 = \underline{43}^2$

9. (i) 9 (ii) 36

10. (i) F (ii) F (iii) F (iv) F
    (v) T (vi) T (vii) T (viii) T

****

EXERCISE 1.2

1. (i) 625 (ii) 1369 (iii) 2916 (iv) 9216 (v) 5041
2. (i) 7921 (ii) 76176 (iii) 121801 (iv) 85849 (v) 25921

3.	(i) 16129	(ii) 55225	(iii) 725904	(iv) 63001	(v) 251001
4	(i) 1225	(ii) 5625	(iii) 9025	(iv) 11025	(v) 42025
5.	(i) 2601	(ii) 2916	(iii) 3136	(iv) 3364	(v) 3481
6.	(i) 259081	(ii) 265225	(iii) 275625	(iv) 336400	(v) 285156
7.	(i) 259081	(ii) 44521	(iii) 390625		
8.	(i) 241081	(ii) 35721	(iii) 330625		

****

EXERCISE 1.3

1.	(ii) and (iv)	**2.** (i) No	(ii) No	(iii) No	(iv) Yes
3.	(i) 1 or 9, odd	(ii) 4 or 6	(iii) 1 or 9, odd	(iv) 5, odd	
4.	(i) 11 and 13				
5.	(i) 27	(ii) 20	(iii) 42	(iv) 64	
6.	(i) 88	(ii) 98	(iii) 77	(iv) 84	
7.	(i) Yes, 44	(ii) Yes, 91			
8.	(i) 5, 30	(ii) 2, 54	(iii) 3, 60	(iv) 7, 84	(v) 3, 78
9.	(i) 5, 6	(ii) 5, 27	(iii) 7, 20	(iv) 11, 64	
10.	49	**11.** 77			

****

EXERCISE 1.4

1.	(i) One	(ii) Two	(iii) Two	(iv) Three	(v) Three
2.	(i) Four	(ii) Four	(iii) Five		
3.	(i) 210	(ii) 165	(iii) 234	(iv) 222	(v) 316
4.	(i) 625	(ii) 345	(iii) 440		
5.	(i) 48	(ii) 67	(iii) 59	(iv) 23	
6.	(i) 38	(ii) 43	(iii) 76	(iv) 89	
7.	(i) 57	(ii) 31	(iii) 40	(iv) 75	
8.	(i) 40	(ii) 110	(iii) 231	(iv) 1176	

****

EXERCISE 1.5

1.	(i) $\frac{19}{25}$	(ii) $\frac{46}{123}$	**2.** (i) $\frac{129}{67}$	(ii) $\frac{333}{555}$	
3.	(i) $4\frac{8}{13}$	(ii) $3\frac{4}{15}$	**4.** (i) $4\frac{23}{27}$	(ii) $7\frac{18}{35}$	
5.	(i) 2.7	(ii) 4.1	(iii) 3.05	(iv) 9.21	
6.	(i) 0.091	(ii) 0.231			
7.	(i) 1.30	(ii) 4.81	(iii) 2.24	(iv) 4.47	(v) 0.32
8.	(i) 0.13	(ii) 0.95	(iii) 2.65	(iv) 0.94	(v) 1.44
9.	(i) 0.025	(ii) 0.645	(iii) 1.416	(iv) 1.049	
10.	(i) F	(ii) F	(iii) F	(iv) T	(v) T

****

EXERCISE 2.1

1.	1, 9, 2, 7, 6, 8, 4, 3, 5				
2.	(i) 42875	(ii) 175616	(iii) 373248	(iv) 64964808	
	(v) 274625000	(iv) 549353259			
3.	(i) 42875	(ii) 175616	(iii) 373248		
4.	(iii)	**5.** (iii) 3	**6.** (iii) 9		
8.	(i) F	(ii) T	(iii) T	(iv) F	(v) F
	(vi) F	(vii) T	(viii) F	(ix) F	(x) F

EXERCISE 2.2

1.	(i) 4	(ii) 8	(iii) 12	
2.	(i) No	(ii) No	(iii) No	(iv) Yes
3.	(i) 5; 5	(ii) 2; 7	(iii) 63; 9	
4.	(i) 1	(ii) 4	(iii) 3	(iv) 6
5.	(i) 6	(ii) 2	(iii) 8	(iv) 5
6.	(i) 73	(ii) 45	(iii) 48	(iv) 36
7.	(i) 63	(ii) 76	(iii) 84	(iv) 85
8.	(i) –61	(ii) –24	(iii) –83	(iv) –56
9.	(i) 135	(ii) 273	(iii) 595	(iv) 385

10. (i) $\frac{9}{13}$ (ii) $\frac{15}{17}$ (iii) $\frac{21}{35}$, *i.e.*, $\frac{3}{5}$ (iv) $\frac{7}{55}$

11. (i) No, 3 (ii) No, 5 (iii) No, 169

12. (i) 9 (ii) 25 (iii) 13

****

EXERCISE 3.1

1. (i) 4 (ii) 3 (iii) 5 2. (i) 2 (ii) 6

3. (i) $\frac{5}{3}$ (ii) $\frac{7}{11}$ 4. (i) $\frac{5}{3}$ (ii) $\frac{7}{11}$

5. (i) $5^{\frac{1}{2}}$ (ii) $7^{\frac{1}{3}}$ (iii) $1100^{\frac{1}{9}}$ (iv) $\left(\frac{3}{4}\right)^{\frac{1}{4}}$ (v) $\left(\frac{61}{1123}\right)^{\frac{1}{8}}$

6. (i) $\sqrt{16}$; 16, 2 (ii) $\sqrt[3]{125}$; 125,3 (iii) $\sqrt[9]{\frac{6}{17}}$; $\frac{6}{17}$; 9

(iv) $\sqrt[11]{\frac{11}{23}}$; $\frac{11}{23}$, 11 (v) $\sqrt[17]{\frac{61}{328}}$; $\frac{61}{328}$, 17

7. (i) 32 (ii) $\frac{27}{8}$ (iii) $\frac{78125}{823543}$ (iv) $\frac{8}{27}$

8. (i) 32 (ii) $\frac{27}{8}$ (iii) $\frac{78125}{823543}$ (iv) $\frac{8}{27}$

9. (i) $\frac{1}{7}$ (ii) $\frac{3}{5}$ 10. (i) $\frac{729}{125}$ (ii) $\frac{243}{32}$

11. (i) 529 (ii) $\frac{1}{1331}$ (iii) 27 (iv) $\frac{1}{3}$

12. (i) 225 (ii) $\left(\frac{13}{2}\right)^{\frac{1}{3}}$ (iii) $\frac{1}{3}$ (iv) $\frac{1}{27}$

13. (i) 0.008 (ii) 0.04 (iii) 15.625 (iv) 0.00032

14. (i) $\frac{1}{5}$ (ii) 2197 (iii) 15 (iv) $\frac{1}{7776}$

15. (i) T (ii) F (iii) F (iv) F (v) T
(vi) T (vii) T

****

EXERCISE 4.1

1. Rs 2400 each
2. Rs 16800
3. Gain : $9\frac{3}{8}\%$
4. (i) Rs 5940 (ii) Rs 5000
5. (i) Rs 825 (ii) Rs 1050 (iii) Loss : $1\frac{11}{25}\%$
6. 2500
7. (i) Rs 3750 (ii) Rs 3375
8. Rs 7500
9. Rs 250000
10. Gain : $14\frac{6}{25}\%$
11. (i) Rs 600 (ii) Rs 624, Rs 630
12. 23.75%
13. Loss : 1%
14. (i) 4 % (ii) Rs 34.56
15. Rs 1592.50, Rs 2012.50

****

EXERCISE 4.2

1. (i) Rs 280 (ii) Rs 891
2. (i) 6% (ii) 20%
3. (i) Rs 2000 (ii) Rs 3000
4. Rs 1261
5. Rs 32200
6. 28%
7. Rs 5500
8. 25%
9. Rs 900
10. Rs 680
11. Rs 1296
12. Rs 3150
13. Rs 2880
14. Rs 1600
15. Rs 937.50

****

EXERCISE 5.1

1. (i) Rs 185.40 (ii) Rs 293.15 (iii) Rs 307.50
(iv) Rs 1050 (v) Rs 1414.40
2. Rs 2125
3. Rs 3310
4. Rs 7493.91
5. Rs 4413. 50
6. Rs 17576; Rs 1951
7. Rs 10360.23
8. Rs 5796
9. Rs 1261
10. Rs 1672.72
11. (i) Rs 112614 (ii) Rs 10810.94

****

EXERCISE 5.2

1. Rs 4410 ; Rs 410
2. Rs 7260 ; Rs 1260
3. Rs 6760 ; Rs 510
4. Rs 24845.94; Rs 4845.94
5. Rs 39366 ; Rs 8116
6. Rs 4775.40
7. Rs 148877
8. Rs 49130
9. Rs 13310
10. Rs 1437.70
11. Fatima by Rs 362.50
12. Rs 43.20
13. Rs 5369
14. Rs 1261
15. Rs 64000
16. Rs 3375
17. Rs 10000
18. Rs 400000
19. Rs 40000
20. 5 %
21. 3 years
22. 2

****

EXERCISE 6.1

1. $x^2 + 9x + 20$
2. $x^2 + 15x + 54$
3. $x^2 + 15x + 56$
4. $x^2 + 13x + 36$
5. $x^2 + 8x + 12$
6. $x^2 + 3x - 4$
7. $p^2 + 2p - 24$
8. $y^2 + 5y - 24$
9. $x^2 - 5x + 4$
10. $z^2 - 15z + 14$
11. $y^2 - 15y + 44$
12. $x^2 + 17x - 84$
13. $x^2 + 5x - 84$
14. $y^2 + 16y - 80$
15. (i) $x^2 + \frac{26}{5}x + 1$ (ii) $y^2 + \frac{77}{12}y + \frac{5}{2}$
16. (i) $z^2 + \frac{25}{12}z + 1$ (ii) $x^4 + 13x^2 + 36$
17. (i) $y^4 + 18y^2 + 72$ (ii) $q^4 + 3q^2 - 4$
18. (i) $p^4 + \frac{63}{4}p^2 - 4$ (ii) $y^4 - \frac{73}{35}y^2 - 2$
19. (i) $z^6 + 15z^3 + 14$ (ii) $z^6 - 7z^3 - 8$
20. (i) $y^6 + \frac{13}{8}y^3 - \frac{3}{4}$ (ii) $x^6 + \frac{77}{136}x^3 - \frac{6}{17}$
21. (i) 10918 (ii) 42228
22. (i) 9120 (ii) 7052
23. (i) 10094 (ii) 9595
24. (i) 36666 (ii) 39360
25. (i) 41382 (ii) 40188

****

EXERCISE 6.2

1. $x^2 + 4y^2 + 16z^2 + 4xy + 16yz + 8xz$
2. $9x^2 + y^2 + 25z^2 - 6xy + 10yz - 30xz$
3. $x^2 + 4y^2 + 36z^2 + 4xy - 24yz - 12xz$
4. $9a^2 + 4b^2 + 9c^2 + 12ab - 12bc - 18ac$
5. $9a^2 + 49b^2 + c^2 - 42ab + 14bc - 6ac$
6. $25a^2 + 49b^2 + c^2 - 70ab - 14bc + 10ac$
7. $16l^2 + 4m^2 + 9n^2 + 16lm - 12mn - 24ln$
8. $4l^2 + m^2 + 64n^2 - 4lm - 16mn + 32ln$
9. $l^2 + 4m^2 + 49n^2 + 4lm - 28mn - 14ln$
10. $p^2 + 81q^2 + 4 + 18pq + 36q + 4p$
11. $36x^2 + \frac{1}{4}y^2 + 16z^2 + 6xy + 4yz + 48xz$
12. $81x^2 + y^2 + \frac{1}{9}z^2 - 18xy - \frac{2}{3}yz + 6xz$
13. $\frac{1}{16}a^2 + \frac{1}{4}b^2 + 256 - \frac{1}{4}ab - 16b + 8a$
14. $a^2 + \frac{1}{4}b^2 + 36 + ab + 6b + 12a$
15. $9x^2 + 16y^2 + 4z^2 - 24xy - 16yz + 12zx$
16. $4x^2 + 9y^2 + 25z^2 + 12xy - 30yz - 20xz$
17. $a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$
18. $a^2 + 4b^2 + 49c^2 - 4ab - 28bc + 14ca$
19. $2p^2 + 2q^2 + 2r^2 + 4qr$
20. $2p^2 + 2q^2 + 2r^2 - 4pq$
21. $4pq + 4pr$
22. $-4qr + 4pr$
23. $8xy + 8xz$
24. $-8xy + 8xz$

****

EXERCISE 6.3

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$
2. $8x^3 - 36x^2y + 54xy^2 - 27y^3$
3. $a^3x^3 + 3a^2x^2by + 3axb^2y^2 + b^3y^3$
4. $x^6 + 6x^4y + 12x^2y^2 + 8y^3$

5. $8x^3 - 12x^2y^2 + 6xy^4 - y^6$

6. $-x^3 + 12x^2y - 48xy^2 + 64y^3$

7. $a^3 + 15a^2y + 75ay^2 + 125y^3$

8. $\frac{1}{27}x^3 + \frac{5}{9}x^2y + \frac{25}{9}xy^2 + \frac{125}{27}y^3$

9. $\frac{1}{27}x^3 - \frac{2}{9}x^2y + \frac{4}{9}xy^2 - \frac{8}{27}y^3$

10. (i) 224 (ii) 1944 (iii) 5833 (iv) $\frac{8001}{8}$

11. (i) 104 (ii) 61 (iii) 559 (iv) –1001

12. (i) 61 (ii) 936 (iii) 2863 (iv) 13897

13. $2a^3 + 24ab^2$

14. $2a^3 + 54ab^2$

15. $250\,b^3 + 120a^2b$

16. $-16\,b^3 - 588\,b$

17. $\frac{2}{27}a^3 + \frac{8}{9}ab^2$

18. $\frac{16}{27}b^3 + \frac{4}{9}a^2b$

19. (i) 1124864 (ii) 1012048064 (iii) 127263527

20. (i) 970299 (ii) 988047936 (iii) 997002999

21. (i) 214921799 (ii) 941.192 (iii) 513.922401

****

EXERCISE 6.4

1. $(x + 1)(x + 9)$

2. $(x + 3)(x + 4)$

3. $(y - 4)(y + 2)$

4. $(y - 7)(y + 1)$

5. $(p + 4)(p - 1)$

6. $(p + 6)(p - 2)$

7. $(m - 5)(m - 3)$

8. $(m - 6)(m - 4)$

9. $(3x + y + 5z)(3x + y + 5z)$

10. $(2x + 3y - 4z)(2x + 3y - 4z)$

11. $(m - 2n + 5z)(m - 2n + 5z)$

12. $(7m - 2n - 3z)(7m - 2n - 3z)$

13. $(3x + y + 5)(3x + y + 5)$

14. $(p + \frac{q}{2} + 1)(p + \frac{q}{2} + 1)$

15. $(\frac{p}{2} + \frac{q}{3} + 6)(\frac{p}{2} + \frac{q}{3} + 6)$

16. $(\sqrt{2}\,x-y+2\sqrt{2}\,z)(\sqrt{2}\,x-y+2\sqrt{2}\,z)$

17. $(\sqrt{3}\,x+\sqrt{3}\,y+z)(\sqrt{3}\,x+\sqrt{3}\,y+z)$ **18.** $(2x + y)(2x + y)(2x + y)$

19. $(2x - y)(2x - y)(2x - y)$ **20.** $(3q - 5p)(3q - 5p)(3q - 5p)$

21. $(4p - 3q)(4p - 3q)(4p - 3q)$

22. $(3 - 5p)(3 - 5p)(3 - 5p)$ **23.** $(4p - 3)(4p - 3)(4p - 3)$

24. $(2x + 9)\,(2x + 9)\,(2x + 9)$ **25.** $(3x - \frac{1}{6})(3x - \frac{1}{6})(3x - \frac{1}{6})$

****

EXERCISE 7.1

1. (ii), (iv), (v), (vi)

2. $4y^4 + y^2 + 6y + 9$; 4 **3.** $-13q^5 + 4q^2 + 12q$; 5

4. $z^2 + \frac{25}{12}z + 1$; 2 **5.** $x^4 + 13x^2 + 36$; 4

6. $-5y^8 + y^2 + 12$; 8 **7.** $4q^8 - q^6 + q^2$; 8

8. $p^7 + p^2 + 16$; 7 **9.** $-\frac{5}{7}y^{11} + y^3 + y^2$; 11

10. $z^6 - 15z^3 + 14$; 6 **11.** $z^6 - 9z^3 + 8$; 6

12. $y^6 + 9y^3 - 22$; 6 **13.** $x^6 + \frac{77}{136}x^3 - \frac{6}{17}$; 6

14. x **15.** $-3x$ **16.** $\frac{2}{3}x$ **17.** $\frac{x}{\sqrt{5}}$ **18.** $\frac{\sqrt{3}}{2}a^2$ **19.** $-\sqrt{2}\,a^2$

20. $\frac{3}{2}x^2 + x + \frac{1}{2}$ **21.** $\frac{1}{3}y^3 - y^2 + \frac{1}{6}y$ **22.** $-2p^2 + 2p + \frac{1}{2} + \frac{2}{p}$

23. $-\frac{x^2}{\sqrt{2}} + \frac{1}{\sqrt{2}}$ **24.** $\frac{5}{2}z^2 - 3z + \frac{7}{2}$ **25.** $\frac{q^2}{\sqrt{3}} + \frac{2}{\sqrt{3}}q$

26. (i) $x + 2$ (ii) $x + 2$ (iii) $y + 3$ (iv) $y - 3$
(v) $z - 3$ (vi) $x^2 + 1$

****

EXERCISE 7.2

1. $3a - 1 ; 9$ **2.** $2b - \frac{1}{5} ; \frac{37}{5}$ **3.** $3p^2 + p - \frac{1}{2} ; \frac{9}{2}$

4. $2q^2 - \frac{5}{2}q + \frac{9}{4}; -\frac{11}{2}$ **5.** $4x^2 - 2 ; 6$

6. $4x^3 + 6x^2 + 2x + \frac{7}{2}; \frac{61}{2}$ **7.** $y^2 - y + 1 ; 0$

8. $z^3 + z - 1 ; 1$ **9.** $x^3 + 5 ; 5$ **10.** $y^2 + 3 ; 6$

12. 0 **13.** 0 **14.** 1 **15.** 1

16. No **17.** Yes **18.** No **19.** No

20. Yes **21.** No **22.** No

****

EXERCISE 8.1

1. $x = -7$ **2.** $x = 3$ **3.** $x = -\frac{21}{2}$ **4.** $x = \frac{35}{33}$

5. $z = -\frac{19}{4}$ **6.** $y = -\frac{13}{4}$ **7.** $y = 1$ **8.** $z = -\frac{3}{11}$

9. $y = \frac{2}{3}$ **10.** $y = 3$ **11.** $k = 4$ **12.** $p = \frac{15}{67}$

13. $x = \frac{7}{12}$ **14.** $x = -\frac{25}{7}$ **15.** $y = -48$ **16.** $z = -\frac{1}{2}$

17. $x = \frac{-118}{39}$ **18.** $y = -4$ **19.** $x = -\frac{8}{33}$ **20.** $x = 10$

21. $x = 2$ **22.** $y = 1$

****

EXERCISE 8.2

1. 12, 48 **2.** 42, 56 **3.** $\frac{13}{21}$ **4.** 15, 45 **5.** 25, 30

6. 216, 222, 228 **7.** 324, 333, 342 **8.** 20 years, 28 years

9. Lovely : 20 years, Luckee : 50 years **10.** $l = 80$ cm, $b = 40$ cm.

11. 36 **12.** 84 **13.** 120 cm **14.** Rs 24000

15. 31.5 km/h **16.** 800 km **17.** 1.5 km/h

****

EXERCISE 9.1

1. (i) Yes, lines parallel to the same line
(ii) Three, AB || EF, EF || DC, DC || AB

2. Six, DE || AB, FG || AB, HI || AB, FG || DE, HI || DE, HI || FG

3. (i) Yes, lines parallel to the same line (ii) 50° **4.** 60°

5. (i) Yes, $\angle A + \angle B = 180^{\circ}$ (ii) Yes, $\angle B + \angle C = 180^{\circ}$

6. (i) Yes, lines perpendicular to the same line (ii) 65°

7. (i) Yes, lines perpendicular to the same line
(ii) Yes, lines parallel to the same line (iii) Yes, from (ii)

8. Three, AB || EF, EF || DC, DC || AB

9. (i) T (ii) F (iii) T (iv) F (v) F (vi) F

****

EXERCISE 9.2

1. (i) Yes, lines parallel to the same line
(ii) Yes, equal intercepts property

2. (i) Yes, equal intercepts property (ii) Yes, AD = AE

3. (i) Yes, equal intercepts property (ii) Yes, equal intercepts property

4. (i) Yes, equal intercepts property (ii) Yes, equal intercepts property

5. (i) Yes, equal intercepts property (ii) Yes, equal intercepts property
(iii) 1.5 cm

6. (i) Yes, it intersects three lines at distinct points

(ii) Yes, it intersects three lines at distinct points

(iii) Yes, lines perpendiculars to the same line

(iv) Yes, equal intercepts property

7. 9 cm 8. 3 cm 9. (i) $\frac{2}{3}$ (ii) 2 cm

10. $\frac{20}{3}$ m, 10 m, $\frac{40}{3}$ m 11. No, equal intercepts property

12. No, proportional intercepts property

****

EXERCISE 9.3

1. 2.5 cm 2. 1.7 cm. 5. 4 cm

****

EXERCISE 10.1

1. Trapezium 3. (i) Rhombus (ii) Rectangle (iii) Square

4. 140°, 140° 5. (i) 18°, 54°, 126°, 162° (ii) Yes, PQ || SR

(iii) No, PS is not parallel to QR

6. (i) T (ii) T (iii) F (iv) T (v) F

(vi) F (vii) T (viii) T (ix) F (x) F

(xi) T (xii) F (xiii) T (xiv) F

****

EXERCISE 10.2

1. 14 cm. 2. 110°, 70, 110° 3. 90° 4. 9 cm, 15 cm, 9 cm, 15 cm

5. 72°, 108°, 72°, 108° 6. 50 cm, 25 cm, 50 cm, 25 cm

7. (i) Yes, opposite sides of a parallelogram

(ii) Yes, opposite sides of a parallelogram

(iii) Yes, same line segment (iv) Yes, SSS.

8. No, diagonals not bisecting each other

9. (i) Diagonals bisect at O (ii) Alternate angles
(iii) Vertically opposite angles (iv) ASA; Yes

10. (i) Opposite angles of a parallelogram (ii) AF bisects $\angle A$
(iii) CE bisects $\angle C$ (iv) From (i), (ii) and (iii)
(v) Alternate angles (vi) From (iv) and (v)
(vii) Corresponding angles equal (viii) $AB \parallel CD$
(ix) Opposite sides are parallel

****

EXERCISE 10.3

1. (i), (ii), (v), (vii) 2. (i), (iii), (iv), (viii), (x)
3. (i), (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix), (x)
4. No, diagonals are not perpendicular
5. (i) Yes, opposite sides of a rectangle
(ii) Yes, opposite sides of a rectangle (iii) Yes, each angle of 90°
(iv) Yes, SAS
6. (i) Yes, diagonals bisect each other (ii) Yes, sides of a rhombus
(iii) Yes, SSS
(iv) Yes, Corresponding Parts of Congruent Triangles (CPCT)
(v) Yes, SSS (vi) Yes, CPCT (vii) Yes, from (iv) and (vi)
7. 120°, 60°, 120°, 60° 8. No, diagonals not equal
9. (i) Yes, opposite sides of a rectangle (ii) Yes, each of 90°
(iii) Yes, alternate angles (iv) Yes, ASA (v) Yes, CPCT
10. No 11. Yes 12. No
13. 13cm, 13cm, 13cm, 13cm; Rhombus 14. Square

****

EXERCISE 11.1

5. No, since AB + AD = BD

****

EXERCISE 11.2

3. No, since BD + AB < AD

****

EXERCISE 11.3

5. No, since $\angle A + \angle B + \angle C > 360^\circ$

****

EXERCISE 12.1

1. 3 cm 2. 24 cm 3. 6 cm 4. 10 cm 5. 5 cm
6. Point of intersection of perpendicular bisectors of AB and BC.
7. (i) Line segment joining the centre and the mid – point of the chord
   (ii) same as in (i)
   (iii) $\angle AMP + \angle BMP = 180^\circ$
8. (i) AB = BC (ii) RHS (iii) CPCT
9. (i) AB = CD (ii) RHS (iii) CPCT
   (iv) MS + AM = NS + NC (v) AB – AS = CD – CS
10. (i) ASA (ii) CPCT (iii) Chords equidistant from the centre
11. (i) T (ii) F (iii) T (iv) T

****

EXERCISE 12.2

1. 120°, 120°, 120°
2. (i) No, central angles not equal (ii) No, central angles not equal
   (iii) Yes, central angles equal (iv) No, central angles not equal
   (v) Yes, central angles equal (vi) No, central angles not equal
   (vii) Yes, central angles equal
3. (i) 60° (ii) 55° (iii) 40° (iv) 240°
4. (i) 90° (ii) Yes 5. (i) 50° (ii) 25°
6. (i) 40° (ii) 50° 7. (i) 15° (ii) 25° (iii) 100° (iv) 50°

8. (i) Yes, a pair of alternate interior angles.

(ii) Yes, angles subtended by an arc at the centre and at the remaining part of the circle

(iii) Yes, same as in (ii) (iv) Yes, from (i), (ii) and (iii)

(v) Yes, central angles equal

9. (i) 160° (ii) 80°

10. (i) 60° (ii) $37\frac{1}{2}^\circ$ (iii) $37\frac{1}{2}^\circ$ (iv) $22\frac{1}{2}^\circ$

****

EXERCISE 12.3

1. 110°, 105° 2. (i) 85° (ii) 115° (iii) 95° (iv) 65° (v) 85° (vi) 115°

3. (i) 70° (ii) 110° 4. (i) 40° (ii) 100° (iii) 70°

5. (i) Interior angles on the same side of a transversal

(ii) ABCD is a cyclic quadrilateral.

(iii) From (i) and (ii)

6. (i) Yes, opposite angles of a parallelogram

(ii) Yes, opposite angles of a cyclic quadrilateral (iii) Yes, from (i) and (ii)

(iv) Yes, same as (iii) (v) Yes, each angle 90°

7. (i) 180° (ii) 180° (iii) 180° (iv) 540° (v) 360°

8. (i) 55° (ii) 100°

****

EXERCISE 13.1

1. $84\ cm^2$ 2. $60\ dm^2$ 3. $285\ cm^2$

4. (i) $12\ m^2$ (ii) $1.24\ m^2$ (iii) $8.1\ m^2$ (iv) $0.0135\ m^2$

5. $24\ cm^2$ 6. $2600\ cm^2$ 7. 4 cm 8. 5 m 9. 15 cm

10. 40 m 11. 12 cm 12. 12.5 cm 13. 15 cm 14. Rs 210

****

EXERCISE 13.2

1. 63 cm^2 **2.** 4500 dm^2 **3.** 180 cm^2

4. (i) 0.6m^2 (ii) 0.155 m^2 (iii) 3.2 m^2 (iv) 67.5 m^2

5. 20 cm **6.** 10 cm **7.** 8 cm **8.** 10 m **9.** 3 m

10. $225\sqrt{3}$ cm^2 **11.** $16\sqrt{3}$ dm^2 **12.** 800 cm^2

13. (i) 120 cm^2 (ii) 2500 cm^2 (iii) 280 cm^2 **14.** 600 m^2

15. 200 m **16.** 441 m^2

17. (i) $\frac{P}{2}$ square units, $\frac{P}{2}$ square units

(ii) $\frac{P}{3}$ square units, $\frac{P}{3}$ square units, $\frac{P}{3}$ square units

(iii) Mark $(n-1)$ points on any side of the triangle to get n equal parts of that side. Join these points with the opposite vertex.

****

EXERCISE 13.3

1. 96 cm^2 **2.** 228 dm^2 **3.** 303 cm^2

4. (i) 1.6 m^2 (ii) 0.0725 m^2 (iii) 28 m^2 (iv) 2.025 m^2

5. 20 cm **6.** 8 cm **7.** $\frac{10}{\sqrt{3}}$ cm **8.** 10 m **9.** 3 m

10. 12 cm, 18 cm **11.** 10 cm, 20 cm **12.** 80 cm^2 **13.** 216 cm^2

****

EXERCISE 13.4

1. (i) 44 cm (ii) $34\frac{4}{7}$ dm (iii) $62\frac{6}{7}$ m

2. (i) 15.7 cm (ii) 9.42 dm (iii) 1.57 m

3. (i) 4 cm (ii) 28 dm (iii) 5 m

4. (i) 1 cm (ii) 350 dm (iii) 49 m

5. $6\frac{2}{7}$ cm	**6.** 12 cm	**7.** 880 m	**8.** 50	**9.** 44 m
10. 264 m	**11.** 3.98 cm (approx.)		**12.** 3 : 4	
13. 30 dm	**14.** 30 cm	**15.** 41.1 m	**16.** 2411520 km	

17. 3.1415929 . . . It is an aproximate value of p.

****

EXERCISE 13.5

1.	(i) 1386 cm^2	(ii) 7546 dm^2	(iii) 147994 cm^2	
2.	(i) $314\frac{2}{7}$ cm^2	(ii) 75.46 dm^2	(iii) $3\frac{1}{7}$ m^2	
3.	(i) 7 cm	(ii) 14 dm	(iii) 63 cm	
4.	(i) 21 cm	(ii) 10 dm	(iii) 25 cm	
5.	(i) 20 cm	(ii) 100 dm	(iii) $20\sqrt{15}$ cm	
6.	$78\frac{4}{7}$ cm^2	**7.** 314.28 m^2	**8.** 31428.57 m^2	
9.	(i) 78.5 m^2	(ii) 235.5 m^2	**10.** 157 m^2	
11.	154 m^2	**12.** Circle	**13.** 492.86 cm^2 (approx.)	
14.	1.7 m^2 (approx.)		**15.** 5 : 6	**16.** 4 : 1

****

EXERCISE 14.1

1. 1760 cm^2	**2.** 1.76 m^2	**3.** 628 cm^2	**4.** 15972 cm^2
5. 1584 m^2	**6.** Rs 68.75	**7.** 0.55 cm	**8.** 1 m
9. (i) 968 cm^2	(ii) 1161.6 cm^2	(iii) 2140.66 cm^2	
10. (i) 110 m^2	(ii) Rs 4400	**11.** (i) 66 m^2	(ii) Rs 1650
12. Rs 63.53			

****

EXERCISE 14.2

1. 220 cm^2	**2.** (i) 198 cm^2	(ii) 352 cm^2	**3.** 4710 cm^2
4. 682 dm^2	**5.** 424.29 cm^2	**6.** (i) 26 m	(ii) Rs 176502.86

7. (i) 8 cm (ii) 729.14 cm^2 8. 62.8 m 9. Rs 1155

10. 47.1 m^2 11. 5500 cm^2 12. 3 cm

****

EXERCISE 14.3

1. (i) 616 cm^2 (ii) 1386 cm^2 (iii) 38.5 m^2

2. (i) 1386 cm^2 (ii) 394.24 m^2 (iii) 2464 cm^2

3. 1 : 4 4. Rs 27.72 5. Rs 2993.76 6. 3.5 cm 7. 1 : 16

8. 616 cm^2 9. 173.25 cm^2 10. (i) $4\pi r^2$ (ii) $4\pi r^2$ (iii) 1 : 1

****

EXERCISE 15.1

1. (i) 2310 cm^3 (ii) 693 cm^3 (iii) 369. 6 m^3 (iv) 38.5 m^3

2. 34.65 l 3. 17.16 kg 4. 49.5 kg

5. (i) 308 m^3 (ii) 0.5 m 6. (i) 770 m^3 (ii) 154 m^2 (iii) 5 m

7. Cylinder, 85 cm^3 8. (i) 3 cm (ii) 141.3 cm^3

9. (i) 110 m^2 (ii) 1.75 m (iii) 96.25 kl 10. 0.4708 m^2

****

EXERCISE 15.2

1. (i) 264 cm^3 (ii) 154 cm^3 2. (i) 1.232 l (ii) $\frac{11}{35}\,l$

3. 314 cm^2 4. 21 cm 5. 8 cm 6. 38.5 kl

7. (i) 48 cm (ii) 50 cm (iii) 2200 cm^2

8. (i) 226 (ii) 90 (iii) 90

9. 100π cm^3 10. 240π cm^3, 5 : 12

****

EXERCISE 15.3

1. (i) $1437\frac{1}{3}$ cm^3 (ii) $179\frac{2}{3}$ dm^3 (iii) 1.05 m^3 (approx.)

2. (i) $11498\frac{2}{3}$ cm^3 (ii) 0.004851 m^3 (iii) 22.458 dm^3 (approx.)

3. (i) 11.5 l (approx.) (ii) 4.851 l (iii) 22.458 l

4. 0.303 l (approx.) 5. 345.39 g (approx.) 6. $\frac{1}{64}$ 7. 0.06348 m^3 (approx.)

8. $179\frac{2}{3}$ cm^3 9. (i) 249.48 m^2 (ii) 523.9 m^3 (approx.)

10. (i) $3r$ (ii) 1 : 9

****

EXERCISE 16.1

1. 6, 11 2. 2780 3. 81.9

4. (i) 151 cm (ii) 128 cm (iii) 23 cm (iv) 142.1 cm (v) 4

5. (i) 25.6 mm (ii) 8.5 mm (iii) 5

6. (i) 10°C (ii) 25.25°C 7. 15

8. (i) 25 minutes (ii) 4 (iii) 23.6 minutes

9. 8 10. 27 11. 18 12. 5.5 13. $\frac{41}{6}$

14. (i) 41.67 (ii) 91 15. 42

16.

Score	48	58	64	66	69	71	73	81	83	84
Frequency	3	3	4	7	6	3	2	1	2	2

17.

Number of members	3	4	5	6	7	8
Frequency	5	6	3	4	1	1

(i) 3 members, 5 (ii) 8 members, 1 (iii) 4 members

18.

Number of Road accidents	0	1	2	3	4	5	6
Frequency	2	3	6	4	4	6	5

19.

Score	1	2	3	4	5	6
Frequency	5	5	4	3	4	4

20.

Weights (in kg)	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56
Frequency	1	2	1	3	1	1	2	2	4	3	1	2	1	2	1	1	2

(i) 40 kg (ii) 1 (iii) 2 (iv) 48 kg

21.

Marks	17	18	19	21	23	24	25	27	28	29	30	31	32	34	35	36	37	38	39	40	41	42	43	46
Frequency	2	2	3	2	2	5	2	3	1	5	1	2	2	2	2	5	4	1	2	2	3	5	1	1

Range : 29

22. (i) 5 (ii) 8 (iii) 5 (iv) 4
(v) 8 (vi) 5

****

EXERCISE 16.2

1.

Class Interval (Marks)	0-5	5-10	10-15	15-20	20-25
Frequency	6	10	7	9	8

2.

Weekly saving (in Rs)	25-30	30-35	35-40	40-45	45-50	50-55	55-60	60-65	65-70	70-75	75-80	80-85
Freque-ncy	1	0	6	5	1	2	3	2	1	3	2	4

3.

Weekly Pocket Expenses (in Rs)	30-40	40-50	50-60	60-70
Frequency	16	14	57	13

4. (i) 60 (ii) 42.5, 47.5, 52.5, 57.5, 62.5 (iii) 5 (iv) 50–55

5.

Maximum Temperature (in °C)	32-33	33-34	34-35	35-36	36-37	37-38	38-39	39-40	40-41	41-42	42-43	43-44
Frequency	4	3	3	6	3	0	3	2	2	2	1	1

Minimum Temperature (in °C)	21-22	22-23	23-24	24-25	25-26	26-27	27-28	28-29	29-30	30-31	31-32	32-33	33-34
Frequency	1	1	5	2	2	2	2	4	3	4	1	2	1

6.

Earnings (in Rs)	500-1000	1000-1500	1500-2000	2000-2500	2500-3000	3000-3500	3500-4000
Number of stores	3	0	16	4	3	2	2

7.

Pulse Rate per Minute	60-65	65-70	70-75	75-80	80-85
Number of persons	5	3	12	7	3

8. (i) It depicts the marks obtained by 43 students of a class in Physics.
(ii) 10 (iii) 3
(iv) Groups 10 – 20, 80 – 90, 90 – 100 and Groups 50 – 60, 70 – 80
(v) 3 (vi) 21

9. (i) It depicts the ages of 26 teachers of a School
(ii) 3 (iii) 2 (iv) 45–50 (v) 35 – 40 (vi) 5
(vii) 22.5, 27.5, 32.5, 37.5, 42.5, 47.5, 52.5 (viii) 6

10.

Marks	0-5	5-10	10-15	15-20	20-25	25-30	30-35	35-40	40-45	45-50	50-55
No. of Students	3	6	4	5	5	4	1	2	2	2	1

(i) 50 (ii) 5–10

11. (i) It depicts the number of literate females of a village in different age groups.
(ii) 15–20 (iii) 45–50 (iv) 5
(v) 12.5, 17.5, 22.5, 27.5, 32.5, 37.5, 42.5, 47.5 (vi) 2700

12. (i) 12 (ii) 4 (iii) 5 (iv) 87.5

13.

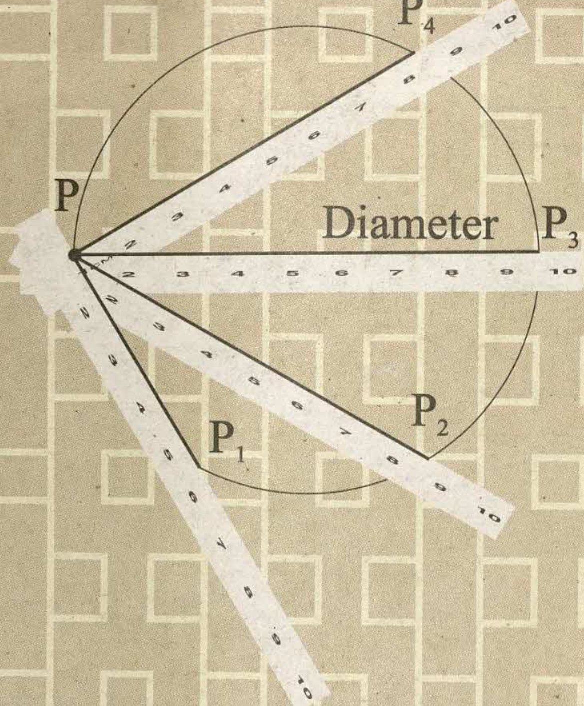
Marks	*Tally Marks*	*Frequency*
0–20		3
20–40		7
40–60	卌 卌 卌 |||	18
60–80		11
80–100	||	2

14. Class Marks : 45, 55, 65, 75, 85

15.

Salary (in thousand Rs)	15-20	20-25	25-30	30-35	35-40
Number of Employees	35	30	45	40	10

****



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